View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by Elsevier - Publisher Connector

Available online at www.sciencedirect.com

ScienceDirect

Nuclear Physics B 907 (2016) 509–541

NUCLEAR PHYSICS B

www.elsevier.com/locate/nuclphysb

Vertex operators of ghost number three in Type IIB supergravity

Andrei Mikhailov [1]

Instituto de Física Teórica, Universidade Estadual Paulista, R. Dr. Bento Teobaldo Ferraz 271, Bloco II – Barra Funda, CEP:01140-070 – São Paulo, Brazil

Received 27 October 2015; accepted 4 April 2016

Available online 11 April 2016

Editor: Herman Verlinde

Abstract

We study the cohomology of the massless BRST complex of the Type IIB pure spinor superstring in flat space. In particular, we find that the cohomology at the ghost number three is nontrivial and transforms in the same representation of the supersymmetry algebra as the solutions of the linearized classical supergravity equations. Modulo some finite dimensional spaces, the ghost number three cohomology is the same as the ghost number two cohomology. We also comment on the difference between the naive and semi-relative cohomology, and the role of b-ghost.

© 2016 Published by Elsevier B.V. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/). Funded by SCOAP^3.

1. Introduction

Vertex operators are one of the central objects in string theory. They represent cohomology classes of the BRST operator. The BRST cohomology depends on the chosen background, and in fact describes the tangent space to the moduli space of backgrounds at the chosen point.

In particular, let us look at the pure spinor superstring theory in expansion around flat space. The structure of massless BRST cohomology in flat space is more or less clear, but it appears

E-mail address: andrei@ift.unesp.br.

[1] On leave from Institute for Theoretical and Experimental Physics, ul. Bol. Cheremushkinskaya, 25, Moscow 117259, Russia.

http://dx.doi.org/10.1016/j.nuclphysb.2016.04.007

0550-3213/© 2016 Published by Elsevier B.V. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/). Funded by SCOAP^3.

that it has never been explicitly spelled out in the literature. The present paper is aimed at filling this gap.

For the closed *bosonic* string the cohomology was computed in [1]. We will here do a similar computation for the pure spinor superstring, but with the following difference. It is well known that the physically relevant cohomology problem is the so-called semirelative cohomology [2], which is $Q_{\rm BRST}$ acting on the vertex operators V satisfying the following condition:

$$(b_0 - \overline{b}_0)V = 0 \tag{1}$$

This condition was built-in into the computations of [1]. In the pure spinor superstring, the construction of the b-ghost is very subtle. In our paper we will compute the "naive" cohomology of $Q_{\rm BRST}$, without taking into account (1). Failure to take into account (1) leads to some strange results:

1. Nonphysical vertex operators, *i.e.* elements of the BRST cohomology which do not correspond to any linearized SUGRA solutions
2. Absence of the dilaton zero mode
3. Nontrivial cohomology at the ghost number three

Problems 1 and 2 are removed if we require the existence of the dilaton superfield Φ (see [3] and the discussion in Section 7.3). To defeat the ghost number three cohomology is more difficult. It is dangerous as a potential obstacle for continuing an infinitesimal solution to a finite solution (*i.e.* obstructed deformations of the flat spacetime). Such obstructions would render the theory physically inconsistent. In bosonic string, all linearized deformations are unobstructed. One explanation is that the *semi-relative* cohomology at the ghost number three is zero, and therefore there is no obstacle. More precisely, the higher order correction to V are controlled by the string field equation [4,5]:

$$QV = (b_0 - \overline{b}_0)(V^2) + \dots \tag{2}$$

Since the ghost number four cohomology is zero, V^2 is in the image of Q. In fact, the pre-image could be chosen to be annihilated by $L_0 - \overline{L}_0$, and this shows that Eq. (2) can be resolved order by order in the deformation parameter.

Unfortunately, we do not have such a proof in the pure spinor formalism. It follows from the consistency of [6] that there is actually no obstacle in extending the infinitesimal deformation to higher orders. Even though the ghost number three cohomology is nonzero, the actual obstruction vanishes for physical states. It would be good to have a transparent proof of this fact using the language of BRST cohomology and vertex operators. This would probably require the use of the composite b-ghost.

1.1. Plan of the paper

In the rest of this introductory section we will review general facts about the BRST cohomology and its relation to the deformations of the worldsheet sigma-model. Then in Section 2 we will review the cohomology of the classical electrodynamics, and explain how to reduce the cohomology of the Type IIB BRST operator in flat space to the cohomology of electrodynamics. The relation will involve the computation of the cohomology of the algebra of translations with coefficients in the space of solutions of SUSY Maxwell equations (Section 3) and the tensor produce of two copies of such spaces (Section 5). The results on BRST cohomology are summarized in Sections 7 and 8.

1.2. Classical sigma-model and its deformations

It was shown in [6] that classical solutions of the Type IIB supergravity are in one-to-one correspondence with two-dimensional sigma-models satisfying certain axioms. Most importantly, there should be two nilpotent odd symmetries Q_L and Q_R:

$$Q_L^2 = Q_R^2 = \{Q_L, Q_R\} = 0 \tag{3}$$

Also, there should be conserved charge known as the "ghost number", with both Q_L and Q_R having ghost number $+1$.

Suppose that we are given such a sigma-model. A natural question is, how can it be deformed? Deformations of the sigma-model are the deformations of the action:

$$S \to S + \varepsilon \int U \tag{4}$$

where U is some operator. If U vanishes on-shell, then such deformation is trivial, as it can be undone by a field redefinition. Suppose that the deformation is nontrivial.

1.3. From integrated vertex to unintegrated vertex

The condition that the deformed action still has a pair of nilpotent symmetries is equivalent to requiring the existence of X_L and X_R such that on-shell:

$$Q_L U \simeq dX_L \quad \texttt{and} \quad Q_R U \simeq dX_R \tag{5}$$

Here $\simeq$ means "equivalent on-shell", *i.e.* "equivalent modulo the equations of motion". Explicitly, (5) implies the existence of infinitesimal transformations q_L and q_R (vector fields on the field space) such that:

$$Q_L U + \varepsilon q_L \mathcal{L} = d\widetilde{X}_L \quad \texttt{and} \quad Q_R U + \varepsilon q_R \mathcal{L} = d\widetilde{X}_R \tag{6}$$

were $\mathcal{L}$ is the sigma-model Lagrangian. (The $\widetilde{X}_{L|R}$ of (6) may be different from the $X_{L|R}$ of (5) because the variation of the Lagrangian is proportional to the equations of motion only modulo a total derivative). Then $Q_L + \varepsilon q_L$ and $Q_R + \varepsilon q_R$ are both symmetries of the deformed action (4). Actually they are nilpotent:

$$(Q_L + \varepsilon q_L)^2 = (Q_R + \varepsilon q_R)^2 = \{Q_L + \varepsilon q_L\ ,\ Q_R + \varepsilon q_R\} = O(\varepsilon^2) \tag{7}$$

This is automatically true because all those anticommutators would be conserved charges of the ghost number two. In this paper we study vertices which are homogeneous polynomials of x and θ. The conserved charges of the ghost number two are polynomials of low degree. Therefore if U is of large enough degree in x and θ, then the nilpotence condition (7) is satisfied.

It is enough to verify (5) for $Q = Q_L + Q_R$:

$$\exists X \ \texttt{such that}\ QU = dX \tag{8}$$

$$\texttt{where}\quad Q = Q_L + Q_R \tag{9}$$

Conditions (5) and (8) are equivalent, because $Q_L U$ and $Q_R U$ are independent, as both left and right ghost number are conserved. In fact, any linear combination $\alpha Q_L + \beta Q_R$ with nonzero constant α and β can be choosen as a BRST operator; all such complexes are quasi-isomorphic to each other.

Operators U satisfying the condition (8) are called "integrated vertices". Notice that X is a one-form of the ghost number one, and $d(QX) = 0$; this typically[2] implies $QX = dV$, because there are no conserved charges of the ghost number two. This V is called the *unintegrated vertex* corresponding to the integrated vertex U:

$$(Q_L + Q_R)X = dV \tag{10}$$

It is also possible to revert this procedure and go from V back to U. This involves the assumption about the vanishing of the cohomology at the nonzero conformal weight.[3] Although (to the best of our knowledge) the proof of this vanishing theorem has never been given, we feel that the statement is true. Notice that the construction of [7] establishes the correspondence between integrated and unintegrated vertices independently of this assumption. Although (in its current form) it only works in flat space and in $AdS_5 \times S^5$, it also teaches us something about the generic curved background. For example, it tells us that the map $U \mapsto V$ is injective. Indeed, suppose that existed an integrated vertex U such that $QU = dX$ and $QX = 0$ (*i.e.* nonzero U gives V). Let us expand such U in Taylor series around a fixed point in the curved space–time, and take the leading term. This should give us the flat space vertex. Since the map $U \mapsto V$ is injective in flat space, the leading term in V should also be nonzero. This means that, if V gets killed, then U cannot survive either.

In any case, our **working hypothesis** is:

- at the linearized level the deformations of the action are in one-to-one correspondence with the BRST cohomology of $Q = Q_L + Q_R$ at the ghost number two

1.4. Ghost number three vertices as obstacles to deformations

If U is an integrated vertex operator, then (4) defines a deformation of the sigma-model action to the first order in ε. It is natural to ask, if the deformation can be continued to higher orders of ε. An obstacle can, in principle, arise already at the order ε^2. Once we deform the action as in (4), the BRST operator gets deformed:

$$Q \to Q + \varepsilon q \tag{11}$$

Here q is such that:

$$QU + q\mathcal{L} = dX \tag{12}$$

where $\mathcal{L}$ is the sigma-model Lagrangian (the existence of such q follows from the fact that QU is a total derivative on-shell, this is in the definition of an integrated vertex operator). Let us consider the following expression: $Q(qX - I_{q^2})$ where I_{q^2} is the Hamiltonian generating q^2:

$$q^2\mathcal{L} = dI_{q^2} \texttt{ on-shell} \tag{13}$$

[2] In this paper we will study vertices which are homogeneous polynomial of x and θ; some of our results are only valid under the assumption that the degree of the polynomial is large enough; exceptions may happen for vertices which do not depend on x.

[3] Going from the deformation of the action to the cohomology of $Q_L + Q_R$ requires the absence of local conserved charges with nonzero ghost number; going back (from V to U) requires the vanishing of the cohomology in the sector with positive conformal dimension.

It was proven in [8] that exists a ghost-number-three operator W such that:

$$Q(qX - I_{q^2}) = dW \tag{14}$$

$$\texttt{with} \quad QW = 0 \tag{15}$$

Moreover, the cohomology class of W is the obstacle for extending the deformation to the order ε^2. The same analysis can be extended to higher orders in ε.

Conclusion: If the BRST cohomology at the ghost number three is zero then any infinitesimal deformation can be continued to a finite deformation, at least as a power series in ε. However, if the BRST cohomology at the ghost number three is nonzero, then there is a potential obstacle.

Comment on the derivation in [8] In [8] we concentrated on the perturbation theory around $AdS_5 \times S^5$, while in the present paper we work in flat space. Some of the assumptions leading to Eq. (14) do not work literally in flat space. For example, conserved charges with nonzero ghost number (besides the BRST charge) do exist in flat space [3]. However, these charges do not depend on x. If we restrict ourselves to the polynomial expressions with large enough degree, then the arguments of [8] do apply.

Another way of looking at the obstacle Suppose that we have an unintegrated vertex operator V of the ghost number two. Suppose that we deform the action as in (4) by *some* integrated operator $\widetilde{U}$ (which is related by the descent procedure to *some other* integrated vertex $\widetilde{V}$). The BRST operator gets deformed: $Q \mapsto Q + \varepsilon\widetilde{q}$. The question is, does V survive such a deformation? In other words it is possible to correct $V \mapsto V + \varepsilon v$ in such a way that $(Q + \varepsilon\widetilde{q})(V + \varepsilon v) = o(\varepsilon^2)$? If the cohomology at the ghost number three is trivial, then this is always possible. Otherwise, further analysis is needed: one has to prove that the ghost number three vertex $\widetilde{q}V$ is Q-exact.

A simpler related phenomenon Similar thing happens at the ghost number one. In flat space, there is a nontrivial cohomology at the ghost number one, corresponding to the global symmetries. However, a generic perturbation of the flat space will kill all this ghost number one cohomology. This is obvious, as generic linearized SUGRA solution does not have any global symmetries. What we want to stress, is the cohomological interpretation of why the ghost number one cohomology gets killed: the existence of the ghost number two cohomology.

1.5. Ghost number three cohomology is nonzero

In this paper we will show that the ghost number three cohomology is nonzero.

The more or less general example of a cohomologically nontrivial ghost number three vertex can be obtained as follows. Let us consider a ghost number two vertex for an exponential linearized solution, for example a Ramond–Ramond excitation:

$$V_2 = e^{(k\cdot x)} \left((\theta_L \Gamma^m \lambda_L)(\theta_L \Gamma_m) + [\lambda_L \theta_L^{\geq 4}] \right)_\alpha P^{\alpha\hat{\beta}} \left((\theta_R \Gamma^m \lambda_R)(\theta_R \Gamma_m) + [\lambda_R \theta_R^{\geq 4}] \right)_{\hat{\beta}} \tag{16}$$

where $P^{\alpha\hat{\beta}}$ is a constant polarization tensor, $\hat{k}P = P\hat{k} = 0$. Suppose that a_m is a constant vector such that $(a \cdot k) \neq 0$. Let us consider:

$$V_3 = \left(a_m(\lambda_L \Gamma^m \theta_L) - a_m(\lambda_R \Gamma^m \theta_R)\right) V_2 \tag{17}$$

Notice that V_3 is BRST closed. We will prove in Section 5.1 that it is not BRST exact.[4] Also notice that $(\lambda_L \Gamma^m \theta_L) - (\lambda_R \Gamma^m \theta_R)$ is the ghost number one unintegrated vertex corresponding to the global conserved charge of translations (the momentum of the string). Vertices of the ghost number three transform in the same representation of the super-Poincare algebra as the linearized SUGRA solutions. (In particular, the obstacle for V_3 to be BRST-exact is in fact the scalar $(k \cdot a)$, so all the polarization is in $P^{\alpha\hat{\beta}}$.)

The integrated vertex corresponding to (17) can be constructed as follows. Let U_2 be the integrated vertex corresponding to V_2. Let j be the conserved current corresponding to $(\lambda_L \Gamma^m \theta_L) - (\lambda_R \Gamma^m \theta_R)$:

$$Qj = d\Big((\lambda_L \Gamma^m \theta_L) - (\lambda_R \Gamma^m \theta_R)\Big) \tag{18}$$

Since U_2 is an integrated vertex, exists a 1-form X such that $QU_2 + q\mathcal{L} = dX$. Let us denote:

$$U_3 = \big((\lambda_L \Gamma^m \theta_L) - (\lambda_R \Gamma^m \theta_R)\big)\, U - j \wedge X \tag{19}$$

We have:

$$\begin{aligned} QU_3 = &-\big((\lambda_L \Gamma^m \theta_L) - (\lambda_R \Gamma^m \theta_R)\big)\, dX - \\ &- j \wedge dV_2 - d\big((\lambda_L \Gamma^m \theta_L) - (\lambda_R \Gamma^m \theta_R)\big) \wedge X \simeq \\ \simeq\ & d\Big(jV_2 - \big((\lambda_L \Gamma^m \theta_L) - (\lambda_R \Gamma^m \theta_R)\big)\, X\Big) \end{aligned} \tag{20}$$

The next step is:

$$Q\left(jV_2 - \big((\lambda_L \Gamma^m \theta_L) - (\lambda_R \Gamma^m \theta_R)\big)\, X\right) = d\Big(\big((\lambda_L \Gamma^m \theta_L) - (\lambda_R \Gamma^m \theta_R)\big)\, V_2\Big) \tag{21}$$

We conclude that U_3 is the integrated vertex operator corresponding to V_3. It is a two-form of the ghost number one.

In this paper we will study *polynomial* vertices, *i.e.* vertices depending on x polynomially. The exponential vertices (16) and (17) are sums of infinitely many polynomial vertices. Indeed, $e^{(k \cdot x)}$ can be decomposed in the Taylor series, and the BRST operator preserves the degree of a polynomial (we assign degree 1 to x and degree $\frac{1}{2}$ to θ and λ). Polynomial vertices are, essentially, harmonic polynomials of x dressed with some appropriate θ-dependence.

A low degree example of a polynomial vertex of the ghost number three has been previously constructed in the revised version of [9]. It is equivalent to the linear term in the expansion of V_3 in powers of x.

1.6. Cohomology at ghost number four and higher is zero

We will prove in Section 7.5 that the pure spinor cohomology is zero at the ghost number four. We have proven in [9] that the pure spinor cohomology is zero at the ghost number greater than four.

This implies that the ghost number three cohomology survives the deformation from flat space–time to generic curved space–time. (However, in the case of a generic curved space–time, there are no ghost number one vertices; therefore the construction of Section 1.5 does not work.)

[4] Notice that $a_m(\lambda_L \Gamma^m \theta_L) + a_m(\lambda_R \Gamma^m \theta_R) = Q(a \cdot x)$, but the relative sign in (17) is minus. With the plus sign it would be $Q((a \cdot x)V_2)$.

1.7. Argument for vanishing of the obstruction based on symmetry

Consider an unintegrated vertex operator V and the corresponding deformation of the sigma-model. Can we extend it to the second order in the deformation parameter? The potential obstacle is the ghost number 3 cohomology class W defined in Eq. (14). It is bilinear in V:

$$W = [\![V, V]\!] \tag{22}$$

We will show that W transforms in the linearized supergravity multiplet (*i.e.* in the same representation as V, modulo some discrete states). The map $V \otimes V \to W$ given by (22) defined by (22) should commute with the action of the supersymmetry, in particular with the translations. Moreover, one can see that:

$$\deg(W) = 2\deg(V) - 2 \tag{23}$$

(For example, for the linear dilaton background analyzed in [3], $V \simeq [\lambda^2\theta^4]$ and $q \simeq \left[\lambda\theta^2\frac{\partial}{\partial\theta}\right]$.) This implies that $[\![V_1, V_2]\!]$ can only be nonzero if either V_1 or V_2 is a low degree polynomial.

It should be possible to complete this argument, which would provide a proof of the vanishing of the obstructions to most of the deformations of the flat space at the second order (but this proof will not work at higher orders).

1.8. Plan of the paper

In Section 2 we explain how to compute the massless BRST cohomology of the Type II SUGRA by relating it to the BRST cohomology of the Maxwell theory using the spectral sequence of a bicomplex. In Sections 3, 4, 5 and 6 we compute the second page of that spectral sequence. In Section 7 we finally compute the spectrum of massless states, and in Section 8 we study the action of supersymmetry on the ghost number three vertices.
For the first reading, we recommend the following sequence:
Section 1 $\longrightarrow$ Section 2 $\longrightarrow$ Section 7.
Then Sections 3, 4, 5 and 6 could be read at the second pass.

2. Type IIB BRST complex vs Maxwell complex

We will compute the cohomology of the Type IIB BRST complex by relating it to the super-Maxwell BRST complex.

2.1. Super-Maxwell BRST complex

The cohomology of the super-Maxwell BRST complex:

$$Q_{\text{SMaxw}} = \lambda^\alpha \left(\frac{\partial}{\partial\theta^\alpha} + \Gamma^m_{\alpha\beta}\theta^\beta \frac{\partial}{\partial x^m} \right) \tag{24}$$

is only nontrivial at the ghost numbers 0 and 1. At the ghost number 0 the cohomology is constants: $V(\theta_L, \theta_R, x) = \text{const}$. At the ghost number 1, the cohomology is in one-to-one correspondence with the solutions of the free Maxwell equation and the free Dirac equation. The vanishing of the cohomology at the ghost numbers two and three is equivalent to the following statements:

1. For any current j_m such that $\partial_m j_m = 0$ always exists the gauge field F_{mn} satisfying $\partial_{[k} F_{lm]} = 0$ and $\partial_m F_{mn} = j_n$
2. For any antichiral spinor ψ exists a chiral spinor ϕ such that $\Gamma^m \partial_m \phi = \psi$
3. For any ρ exists j_m such that $\partial_m j_m = \rho$

Example: Let us look at the ghost number two cohomology. The leading term in the θ-expansion is either $(\theta\Gamma^m\lambda)(\theta\Gamma^n\lambda)(\theta\Gamma_{mn}\psi(x))$ or $(\theta\Gamma^m\lambda)(\theta\Gamma^n\lambda)(\theta\Gamma_{mnl}\theta)A^l(x)$. Let us for example investigate the first possibility. The following expression is in the cohomology of $\lambda^\alpha \frac{\partial}{\partial\theta^\alpha}$:

$$(\theta\Gamma^m\lambda)(\theta\Gamma^n\lambda)(\theta\Gamma_{mn}\psi(x)) \tag{25}$$

Now let us study the effect of the $\frac{\partial}{\partial x}$-term in (24). For (25) to survive the action of $(\lambda\Gamma^m\theta)\frac{\partial}{\partial x^m}$ we need:

$$(\lambda\Gamma^l\theta)\frac{\partial}{\partial x^l}(\theta\Gamma^m\lambda)(\theta\Gamma^n\lambda)(\theta\Gamma_{mn}\psi(x)) = \lambda^\alpha \frac{\partial}{\partial\theta^\alpha}\,\texttt{(something)} \tag{26}$$

The “something” on the right hand side always exists, because any expression of the form $[\lambda^3\theta^4]$ annihilated by $\lambda^\alpha \frac{\partial}{\partial\theta^\alpha}$ is automatically in the image of $\lambda^\alpha \frac{\partial}{\partial\theta^\alpha}$. It remains to investigate the possibility of (25) being Q-exact:

$$\begin{aligned}(\theta\Gamma^m\lambda)(\theta\Gamma^n\lambda)(\theta\Gamma_{mn}\psi(x)) &= \\ = (\lambda\Gamma^l\theta)\frac{\partial}{\partial x^l}\Big((\theta\Gamma^k\lambda)(\theta\Gamma_k\phi(x)) &+ \texttt{(terms of higher orders in }\theta\texttt{)}\Big) + \\ + \lambda^\alpha \frac{\partial}{\partial\theta^\alpha}\,\texttt{(something)}\end{aligned} \tag{27}$$

This is possible iff $\psi(x) = \Gamma^m \frac{\partial}{\partial x^m}\phi(x)$. But for any $\psi(x)$ we can find $\phi(x)$ such that $\psi(x) = \Gamma^m \frac{\partial}{\partial x^m}\phi(x)$. This implies that any expression of the type (25) is always BRST-trivial. The class with the leading term $(\theta\Gamma^m\lambda)(\theta\Gamma^n\lambda)(\theta\Gamma_{mnl}\theta)A^l(x)$ is analyzed similarly.

Conclusion:

$$H^0(Q_{\mathrm{SMaxw}}) = \mathbf{C} \tag{28}$$

$$H^1(Q_{\mathrm{SMaxw}}) = \mathrm{Maxwell} \bigoplus \mathrm{Dirac} \tag{29}$$

$$H^{>1}(Q_{\mathrm{SMaxw}}) = 0 \tag{30}$$

Here “Maxwell$\bigoplus$Dirac” stands for the direct sum of the space of solutions of the Maxwell equations and the space of solutions of the Dirac equation.

We now want to relate the super-Maxwell complex to the Type IIB SUGRA complex.

Comment in the revised version It is possible to modify the definition of the BRST complex by imposing the constraint that the cochains are annihilated by $L_0 + \overline{L}_0$. In this case $H^2(Q_{\mathrm{SMaxw}})$ is nonzero and in fact isomorphic (perhaps modulo some zero modes) to $H^1(Q_{\mathrm{SMaxw}})$ — see the recent work [10] and references there. We do not impose any such constraints. Therefore our BRST complex has $H^2(Q_{\mathrm{SMaxw}}) = 0$ for open strings. But for closed strings, we still get the massless $H^3(Q_{\mathrm{SUGRA}})$ nonzero (and isomorphic to $H^2(Q_{\mathrm{SUGRA}})$ up to zero modes).

2.2. Definition of the doubled complex

Let us consider the tensor product of two SMaxwell complexes:

$$Q_{\text{SMaxw}\otimes\text{SMaxw}} = \lambda_L^\alpha \left(\frac{\partial}{\partial \theta_L^\alpha} + \Gamma^m_{\alpha\beta}\theta_L^\beta \frac{\partial}{\partial x_L^m} \right) + \lambda_R^{\hat\alpha} \left(\frac{\partial}{\partial \theta_R^{\hat\alpha}} + \Gamma^m_{\hat\alpha\hat\beta}\theta_R^{\hat\beta} \frac{\partial}{\partial x_R^m} \right) \tag{31}$$

The operator $Q_{\text{SMaxw}\otimes\text{SMaxw}}$ acts on the space of functions $F(\lambda_L, \lambda_R, \theta_L, \theta_R, x_L, x_R)$. We will denote Q_L and Q_R the two terms on the right hand side of (31). This is the "doubled" BRST complex. The difference with the Type IIB SUGRA BRST complex is the splitting $x = x_L + x_R$. In the Type IIB BRST complex there is no separation of x into x_L and x_R:

$$Q_{\text{SUGRA}} = \lambda_L^\alpha \left(\frac{\partial}{\partial \theta_L^\alpha} + \Gamma^m_{\alpha\beta}\theta_L^\beta \frac{\partial}{\partial x^m} \right) + \lambda_R^{\hat\alpha} \left(\frac{\partial}{\partial \theta_R^{\hat\alpha}} + \Gamma^m_{\hat\alpha\hat\beta}\theta_R^{\hat\beta} \frac{\partial}{\partial x^m} \right) \tag{32}$$

The difference with (31) is that the left and the right parts have a common x instead of separate x_L and x_R; the operator Q_{SUGRA} acts on the space of functions $F(\lambda_L, \lambda_R, \theta_L, \theta_R, x)$.

The computation of the cohomology of (31) is straightforward, because it is just the tensor product of two Maxwell complexes (24); therefore the cohomology is:

$$H^n(Q_{\text{SMaxw}\otimes\text{SMaxw}}) = \bigoplus_{p+q=n} H^p(Q_{\text{SMaxw}}) \otimes H^q(Q_{\text{SMaxw}}) \tag{33}$$

where the spaces $H^p(Q_{\text{SMaxw}})$ are given by Eqs. (28), (29) and (30).

2.3. Spectral sequence $\mathcal{E}_r^{p,q}$

To compute the cohomology of (32), we relate it to the cohomology of (31) by the following trick. Let us introduce a formal fermionic variable c^m and the operator:

$$Q_{\text{Lie}} = c^m \left(\frac{\partial}{\partial x_L^m} - \frac{\partial}{\partial x_R^m} \right) \tag{34}$$

(We call it Q_{Lie} because it can be thought of as the cohomology of the abelian Lie algebra of translations.) Let us consider the bicomplex:

$$Q_{\text{tot}} = Q_L + Q_R + Q_{\text{Lie}} \tag{35}$$

Consider two ways of computing the cohomology of Q_{tot}. We can either compute first the cohomology of Q_{Lie}, and then consider $Q_L + Q_R$ as a perturbation. Or, first compute $H(Q_L + Q_R)$ and then act on it by Q_{Lie}. This means that there are two different spectral sequences, both converging to $H(Q_{\text{tot}})$.

First Q_{Lie}, then $Q_L + Q_R$: Because of the Poincare lemma, the cohomology of Q_{Lie} is only nontrivial in the ghost number 0, and is represented by the functions $f(\lambda_L, \lambda_R, \theta_L, \theta_R, x_L + x_R)$. Therefore the Type IIB BRST complex is equivalent to the cohomology of $Q_L + Q_R$ acting on the cohomology of Q_{Lie}:

$$H(Q_{\text{SUGRA}}) = H(Q_L + Q_R\ ,\ H(Q_{\text{Lie}})) = H(Q_{\text{tot}}) \tag{36}$$

First $Q_L + Q_R$, then Q_{Lie}: now let us first compute the cohomology of $Q_L + Q_R$, and then consider Q_{Lie} as a perturbation. The resulting spectral sequence will be denoted $\mathcal{E}_r^{p,q}$. It computes the cohomology of the SUGRA BRST complex:

$$\mathcal{E}_1^{p,q} = H^p\left(Q_{\text{Lie}}\,,\ \bigoplus_{q_L+q_R=q} H^{q_L}(Q_L)\otimes H^{q_R}(Q_R)\right) \tag{37}$$

$$\mathcal{E}_r^{p,q} \Rightarrow_p \mathcal{E}_\infty^{p,q} \tag{38}$$

$$\bigoplus_{p+q=n} \mathcal{E}_\infty^{p,q} = H^n(Q_{\text{SUGRA}}) \tag{39}$$

Therefore, the only nontrivial components are:

$$\mathcal{E}_1^{p,0} = \Lambda^p \mathbf{C}^{10} \tag{40}$$

$$\mathcal{E}_1^{p,1} = H^p(Q_{\text{Lie}}\,,\ \text{SMaxw}_L \bigoplus \text{SMaxw}_R) \tag{41}$$

$$\mathcal{E}_1^{p,2} = H^p(Q_{\text{Lie}}\,,\ \text{SMaxw}_L \otimes \text{SMaxw}_R) \tag{42}$$

All other components are zero. The only potentially nonzero differentials are:

$$\mathcal{E}_1^{p,0} \xrightarrow{d_1} \mathcal{E}_1^{p+1,0}\,,\ \mathcal{E}_1^{p,1} \xrightarrow{d_1} \mathcal{E}_1^{p+1,1}\,,\ \mathcal{E}_1^{p,2} \xrightarrow{d_1} \mathcal{E}_1^{p+1,2} \tag{43}$$

$$\mathcal{E}_2^{p,2} \xrightarrow{d_2} \mathcal{E}_2^{p+2,1}\,,\ \mathcal{E}_2^{p,1} \xrightarrow{d_2} \mathcal{E}_2^{p+2,0} \tag{44}$$

$$\mathcal{E}_3^{p,2} \xrightarrow{d_3} \mathcal{E}_3^{p+3,0} \tag{45}$$

$H^0(\text{SM}_L \otimes \text{SM}_R)$	$H^1(\text{SM}_L \otimes \text{SM}_R)$	$H^2(\text{SM}_L \otimes \text{SM}_R)$	$H^3(\text{SM}_L \otimes \text{SM}_R)$	$H^4(\text{SM}_L \otimes \text{SM}_R)$
$H^0(\text{SM}_L \bigoplus \text{SM}_R)$	$H^1(\text{SM}_L \bigoplus \text{SM}_R)$	$H^2(\text{SM}_L \bigoplus \text{SM}_R) = \Lambda^4\mathbf{C}^{10} \bigoplus \mathbf{C} \bigoplus \bigoplus \Lambda^4\mathbf{C}^{10} \bigoplus \mathbf{C}$	$H^3(\text{SM}_L \bigoplus \text{SM}_R) = \Lambda^5\mathbf{C}^{10} \bigoplus \Lambda^5\mathbf{C}^{10}$	$H^4(\text{SM}_L \bigoplus \text{SM}_R)$
$\mathbf{C}$	$\mathbf{C}^{10}$	$\Lambda^2\mathbf{C}^{10}$	$\Lambda^3\mathbf{C}^{10}$	$\Lambda^4\mathbf{C}^{10}$

Therefore, in order to compute the BRST cohomology of SUGRA, we have to:

- first compute the cohomology of Q_{Lie} with coefficients in spaces of solutions of the classical electrodynamics and their tensor products
- then compute the differentials d_r

The first step will be elaborated in Sections 3, 4 and 5, and the second in Section 7.

The reader may want to **skip to Section 7** and return here later.

3. Cohomology of classical electrodynamics

In the previous section we related the cohomology of the SUGRA complex to the Lie algebra cohomology of the algebra of translations $\mathbf{R}^{10}$ with coefficients in the tensor product of solutions of Maxwell and Dirac equations. In order to compute it, we will first compute the cohomology with coefficients in the single space of solutions of Maxwell and Dirac equations. Then, in the next section, we will proceed to compute the cohomology with coefficients in the tensor product of two such spaces.

3.1. Cohomology of $\mathbf{R}^{10}$ with values in solutions of Maxwell equations

Consider the space of solutions of the vacuum Maxwell equations:

$$\frac{\partial}{\partial x^m}\frac{\partial}{\partial x^{[m}}A_{n]} = 0 \tag{46}$$

depending on a parameter c^m, a free Grassmann variable. We need to calculate the cohomology of the operator $c^m \frac{\partial}{\partial x^m}$ acting on this space.

We will start by computing the cohomology of divergenceless currents. Consider the space J of one-forms $j_m(x,c)dx^m$ satisfying $\frac{\partial}{\partial x^m} j_m(x,c) = 0$. This is a subspace of the space of all 1-forms Ω^1:

$$0 \to J \xrightarrow{\subset} \Omega^1 \xrightarrow{\delta} \Omega^0 \to 0 \tag{47}$$

This gives the long exact sequence of cohomology:

$$0 \to \mathbf{C}^{10} \to \mathbf{C}^{10} \to \mathbf{C} \to H^1(J) \to 0 \to 0 \to H^2(J) \to 0 \to \dots \tag{48}$$

We conclude:

$$H^0(J) = \mathbf{C}^d \tag{49}$$

$$H^1(J) = \mathbf{C} \tag{50}$$

$$H^{>1}(J) = 0 \tag{51}$$

Now we proceed to the cohomology of the Maxwell complex. A solution of the Maxwell equation is completely characterized by its curvature. The space of solutions is therefore the same as the space of closed 2-forms $F_{mn}dx^m \wedge dx^n$ satisfying $\partial^m F_{mn} = 0$. It is included in the following short exact sequence:

$$0 \to F \to Z^2 \to J \to 0 \tag{52}$$

where Z^2 is the space of all closed 2-forms. The corresponding long exact sequence reads:

$$\begin{aligned}
&\longrightarrow \Lambda^2\mathbf{C}^d \longrightarrow \Lambda^2\mathbf{C}^d \longrightarrow \mathbf{C}^d \longrightarrow \\
&\longrightarrow H^1(\text{Maxwell}) \longrightarrow H^1(Z^2) \longrightarrow \mathbf{C} \longrightarrow \\
&\longrightarrow H^2(\text{Maxwell}) \longrightarrow H^2(Z^2) \longrightarrow 0 \longrightarrow \dots
\end{aligned} \tag{53}$$

To calculate the cohomology of Z^2 we use:

$$0 \longrightarrow Z^1 \longrightarrow \Omega^1 \longrightarrow Z^2 \longrightarrow 0 \tag{54}$$

and

$$0 \longrightarrow \mathbf{C} \longrightarrow \Omega^0 \longrightarrow Z^1 \longrightarrow 0 \tag{55}$$

This implies that for $k > 0$: $H^k(Z^2) = H^{k+1}(Z^1) = H^{k+2}(\mathbf{C}) = \Lambda^{k+2}\mathbf{C}^d$. Therefore, we obtain from (53):

$$H^0(\text{Maxwell}) = \Lambda^2\mathbf{C}^d \;:\; f_{[mn]}dx^m \wedge dx^n \tag{56}$$

$$H^1(\text{Maxwell}) = \mathbf{C}^d \oplus \Lambda^3\mathbf{C}^d \;:\; c_k f_l dx^k \wedge dx^l \;\texttt{and}\; f_{[klm]}c^k dx^l \wedge dx^m \tag{57}$$

$$H^2(\text{Maxwell}) = \mathbf{C} \oplus \Lambda^4\mathbf{C}^d \;:\; c_k c_l dx^k \wedge dx^l \;\texttt{and}\; f_{[ijkl]}c^i c^j dx^k \wedge dx^l \tag{58}$$

$$H^{n>2}(\text{Maxwell}) = \Lambda^{n+2}\mathbf{C}^d \;:\; f_{[j_1\cdots j_{n+2}]}c^{j_1}c^{j_2}\cdots c^{j_n}dx^{j_{n+1}} \wedge dx^{j_{n+2}} \tag{59}$$

Notice that all these cohomology classes are represented by the *constant* field strength. In other words, the *dilatation symmetry* $x^m\frac{\partial}{\partial x^m}$ acts as zero in cohomology.

3.2. Cohomology of $\mathbf{R}^{10}$ with values in solutions of Dirac equations

Let $\mathcal{D}$ denote the space of solutions of the Dirac equations, and $\mathcal{S}$ the space of chiral–spinor-valued functions, and $\mathcal{S}^*$ the antichiral–spinor-valued functions. There is a short exact sequence:

$$0 \to \mathcal{D} \xrightarrow{\subset} \mathcal{S} \xrightarrow{\Gamma^m\frac{\partial}{\partial x^m}} \mathcal{S}^* \to 0 \tag{60}$$

This leads to the long exact sequence of the cohomologies:

$$0 \to \mathbf{C}^{16} \to \mathbf{C}^{16} \xrightarrow{0} \mathbf{C}^{16} \to H^1(\mathcal{D}) \to 0 \to 0 \to H^2(\mathcal{D}) \to 0 \to \cdots \tag{61}$$

Therefore:

$$H^0(\text{Dirac}) = \mathbf{C}^{16} \;:\; \texttt{constant spinors} \tag{62}$$

$$H^1(\text{Dirac}) = \mathbf{C}^{16} \;:\; \hat{c}\Psi \;\texttt{where}\; \Psi \;\texttt{is constant} \tag{63}$$

$$H^{n>1}(\text{Dirac}) = 0 \tag{64}$$

4. Zeroth cohomology of the tensor product of two classical electrodynamics

This is the direct sum:

$$\begin{aligned} &H^0(\text{Maxw}_L \otimes \text{Maxw}_R) \oplus H^0(\text{Dirac}_L \otimes \text{Dirac}_R) \oplus \\ &\oplus H^0(\text{Maxw}_L \otimes \text{Dirac}_R) \oplus H^0(\text{Dirac}_L \otimes \text{Maxw}_R) \end{aligned} \tag{65}$$

The space $H^0(\text{SMaxw}_L \otimes \text{SMaxw}_R)$ can be thought of as the space of functions:

$$F_{[mn]\,;\,[pq]}(x) \tag{66}$$

satisfying:

$$\partial_{[k}F_{mn]\,;\,[pq]} = 0 \tag{67}$$

$$F_{[mn]\,;\,[pq}\overleftarrow{\partial}_{r]} = 0 \tag{68}$$

$$\partial^m F_{[mn]\,;\,[pq]} = 0 \tag{69}$$

$$F_{[mn]\,;\,[pq]}\overleftarrow{\partial}^q = 0 \tag{70}$$

Eqs. (67) and (68) together imply that:

$$F_{[mn\,;\,pq]} = \text{const} \tag{71}$$

$$g^{mp}g^{nq}F_{[mn]\,;\,[pq]} = \text{const} \tag{72}$$

We can write:

$$F_{[mn]\,;\,[pq]} = \partial_{[m} A^L_{n]\,;\,[pq]} = A^R_{[mn]\,;\,[p} \overleftarrow{\partial}_{q]} \tag{73}$$

A consequence of Eqs. (67), (68), (69), (70) is the existence of ϕ^R_q and ϕ^L_m such that:

$$\partial^m A^R_{[mn]\,;\,p} = \partial_p \phi^R_n \tag{74}$$

$$A^L_{m\,;\,[pq]} \overleftarrow{\partial}^q = \partial_m \phi^L_p \tag{75}$$

This implies:

$$g^{np} F_{[mn]\,;\,[pq]}(x) = \frac{1}{2}\partial_q \left(g^{np} A^R_{[mn]\,;\,p} + \phi^R_m \right) = \frac{1}{2}\partial_m \left(g^{np} A^L_{n\,;\,[pq]} + \phi^L_q \right) \tag{76}$$

Let us denote:

$$B^R_m = g^{np} A^R_{[mn]\,;\,p} + \phi^R_m\,, \tag{77}$$

$$B^L_q = g^{np} A^L_{n\,;\,[pq]} + \phi^L_q \tag{78}$$

In particular:

$$\partial_m B^L_q = \partial_q B^R_m \tag{79}$$

Although $A^R_{[mn];p}$ and ϕ^R_n are only defined by (74) up to:

$$A^R_{[mn];p} \mapsto A^R_{[mn];p} + \partial_p \chi^R_{mn} \tag{80}$$

$$\phi^R_n \mapsto \phi^R_n + \partial^m \chi_{mn}\,, \tag{81}$$

this ambiguity does not affect the definition of B^R_m (and similarly B^L_q). Notice that:

$$\partial_{[m} B^R_{n]} = -\partial_{[p} B^L_{q]} = \text{const} \tag{82}$$

$$\partial^p B^L_p = \partial^p B^R_p = \text{const} \tag{83}$$

Let us denote:

$$B^L_m \pm B^R_m = A^\pm_m \tag{84}$$

Then:

$$\partial_{[q} A^+_{m]} = 0 \tag{85}$$

$$\partial_{(q} A^-_{m)} = 0 \tag{86}$$

The physical meaning of $A^\pm_m$ will be explained in Section 7.2.3.

5. First cohomology of the tensor product of two classical electrodynamics

Having computed the cohomology of Q_{Lie} with values in Maxwell and Dirac solutions, we will now use it to compute the cohomology with values in the tensor product $\text{SMaxw}_L \otimes \text{SMaxw}_R$. Again, we will use some spectral sequence. In order to distinguish it from the spectral sequence of Section 2, we will use the notation[5] $E^{p,q}_r$ (that other one was denoted $\mathcal{E}^{p,q}_r$).

[5] Unfortunately, because of certain limitations of LaTeX, we can not afford similar notations for the differentials d_r.

5.1. Dirac–Dirac sector

5.1.1. Spectral sequence $E_r^{p,q}$

The following group is part of the ghost number 3 cohomology:

$$H^1\left(Q_{\mathrm{Lie}}\;,\;\mathrm{Dirac}\otimes\mathrm{Dirac}\right) \tag{87}$$

In this section we will calculate this cohomology group.

The differential Q_{Lie} is realized on the space of bispinors $P^{\alpha\dot\beta}(x_L,x_R,c)$ satisfying:

$$\frac{\partial}{\partial x_L^m}\Gamma^m_{\alpha\alpha'}P^{\alpha'\dot\beta}(x_L,x_R,c)=0 \tag{88}$$

$$\frac{\partial}{\partial x_R^m}P^{\alpha\dot\beta'}(x_L,x_R,c)\Gamma^m_{\dot\beta'\dot\beta}=0 \tag{89}$$

The differential Q_{Lie} acts as follows:

$$Q_{\mathrm{Lie}}P^{\alpha\dot\beta}=c^m\left(\frac{\partial}{\partial x_L^m}-\frac{\partial}{\partial x_R^m}\right)P^{\alpha\dot\beta} \tag{90}$$

Let us introduce the filtration by the degree N:

$$N=\frac{1}{2}\left(c\frac{\partial}{\partial c}+x_L\frac{\partial}{\partial x_L}-x_R\frac{\partial}{\partial x_R}\right) \tag{91}$$

Then $c^m\frac{\partial}{\partial x_L^m}$ is the leading (of degree zero) term in Q_{Lie} and $-c^m\frac{\partial}{\partial x_R^m}$ is subleading (of degree one). Let us calculate the cohomology of Q_{Lie} using the spectral sequence of this filtration. The first page $E_1^{p,q}$ is:

$$E_1^{p,q}=H^{p+q}\left(c^m\frac{\partial}{\partial x_L^m}\;,\;\frac{F^p(\mathrm{Dirac}\otimes\mathrm{Dirac})}{F^{p+1}(\mathrm{Dirac}\otimes\mathrm{Dirac})}\right) \tag{92}$$

$$d_1=-\,c^m\frac{\partial}{\partial x_R^m}\;:\;E_1^{p\;,\;q}\longrightarrow E_1^{p+1\;,\;q} \tag{93}$$

where F^p consists of polynomials with $N\geq p$. Schematically, $E_1^{p,q}$ consists of expressions of the form

$$P^{\alpha\dot\beta}=[c^{p+q}x_L^{n+p}x_R^{n+q}] \tag{94}$$

satisfying both left and right Dirac equations, representing the cohomology of $c^m\frac{\partial}{\partial x_L^m}$. Just to remember:

$$E_1^{\frac{1}{2}(\#c+\#x_L-\#x_R)\;,\;\frac{1}{2}(\#c+\#x_R-\#x_L)} \tag{95}$$

where $\#x$ means “degree in x“.

Because of Section 3.1, the cohomology of $c^m\frac{\partial}{\partial x_L^m}$ is localized on $n+p=0$, and either $p+q=0$ or $p+q=1$. This means that the only nontrivial components of $E_1^{p,q}$ are the ones represented by the following expressions:

$$E_1^{-m,m}\;:\;P\langle x_R^{\otimes 2m}\rangle \tag{96}$$

$$E_1^{-m+1,m}\;:\;\hat cR\langle x_R^{\otimes(2m-1)}\rangle \tag{97}$$

Here, as usual, we denote $\hat c=c^m\Gamma_m$.

The only nontrivial differential is $d_1 : E_1^{-m,m} \to E_1^{-m+1,m}$. The cohomology of this differential is $E_2^{p,q}$. Notice that $d_2 = 0$. Indeed, the construction of $d_2 : E_2^{p,q} \to E_2^{p+2,q-1}$ involves the inversion of $c^m \frac{\partial}{\partial x_L^m}$ and therefore any expression in the image of d_2 is necessarily in the image of $x_L^m \frac{\partial}{\partial x_L^m}$. But $x_L^m \frac{\partial}{\partial x_L^m}$ acts as zero on E_1 and therefore also on E_2.

Therefore our spectral sequence converges at the second page: $E_2 = E_\infty$.

5.1.2. *The image of* $d_1(E_1^{-m,m})$

The condition that the cohomology class of an expression of the form (97) is cancelled by the d_1 of an expression of the form (96) is:

$$R = -\frac{1}{10}\Gamma^m \frac{\partial}{\partial x_R^m} P \tag{98}$$

$$\texttt{with} \quad \frac{\partial}{\partial x_R^m} P \Gamma^m = 0 \tag{99}$$

Indeed, for any $P(x_R)$ solving the right Dirac equation (99) we can tautologically write:

$$\hat{c}R = -c^m \frac{\partial}{\partial x_R^m} P + c^n \frac{\partial}{\partial x_L^n} \left(\widehat{x}_L R + x_L^m \frac{\partial}{\partial x_R^m} P \right) \tag{100}$$

Then (98) is the necessary and sufficient condition that $\Psi := \widehat{x}_L R + x_L^m \frac{\partial}{\partial x_R^m} P$ satisfies both $\frac{\partial}{\partial x_R^m} \Psi \Gamma^m = 0$ *and* $\Gamma^m \frac{\partial}{\partial x_L^m} \Psi = 0$. (And, moreover, any presentation of $\hat{c}R$ as $-c^m \frac{\partial}{\partial x_R^m} P$ plus $c^m \frac{\partial}{\partial x_L^m}$(smth) will necessarily be of the form (100).)

Comment Those P which satisfy $\Gamma^m \frac{\partial}{\partial x_R^m} P = 0$ are in the kernel of d_1, and therefore they form $E_2^{-m,m}$. They are in the ghost number two cohomology (the Ramond–Ramond fields). We have previously explained that d_2 is zero; if it were not zero, it would have killed the ghost number two cohomology.

Notice that any P satisfying (98) and (99) is automatically harmonic: $\Delta P = 0$, therefore (98) and (99) imply that R satisfies the left Dirac equation:

$$\Gamma^m \frac{\partial}{\partial x_R^m} R = 0 \tag{101}$$

This means that:

$$\Gamma^m \frac{\partial}{\partial x_R^m} R \quad \texttt{is an obstacle for the triviality of} \quad R \tag{102}$$

In the rest of this section we will prove that this is the only obstacle, *i.e.* any R satisfying (101) can be represented as (98), (99).

5.1.3. *Proof that (102) is the only obstacle to the triviality of R*

In this section we will prove that if R is a polynomial of nonzero degree (*i.e.* not a constant), than (102) is the only obstacle to the triviality of R.

Notice that it is always possible to solve for P to satisfy (98), but P will not necessarily satisfy (99). But if the Dirac equation (101) is satisfied, then we have:

$$\Gamma^m\, \partial_m \partial_n P\, \Gamma_n = 0 \tag{103}$$

$$\Delta P = 0 \tag{104}$$

We will now prove that (103) and (104) imply that exist P_L and P_R such that:

$$P = P_L + P_R \tag{105}$$

$$\texttt{where } \Gamma^m \frac{\partial}{\partial x_R^m} P_L = 0 \texttt{ and } \frac{\partial}{\partial x_R^m} P_R \Gamma^m = 0 \tag{106}$$

This implies that P can be chosen to satisfy the right Dirac equation, and therefore R is in the image of d_1.

Proof. Let us switch from the bispinor notations to the forms notations. The left Dirac operator corresponds to $\mathcal{D}_L = d + \delta$ while the right Dirac operator is $\mathcal{D}_R = (-1)^{F+1}(d - \delta)$. Eq. (104) implies that $(\delta d + d\delta)P = 0$ while Eq. (103) implies that $(\delta d - d\delta)P = 0$. Therefore we have:

$$d\delta P = \delta d P = 0 \tag{107}$$

We will now prove that under the condition (107) exist P_L and P_R such that:

$$\begin{aligned} P &= P_L + P_R \\ \mathcal{D}_L P_L &= \mathcal{D}_R P_R = 0 \end{aligned} \tag{108}$$

It is useful to keep in mind the cohomology of the de Rham d on harmonic forms is:

$$H^0(d, \ker\Delta) = H^1(d, \ker\Delta) = \mathbf{C} \quad , \quad H^{>1}(d, \ker\Delta) = 0 \tag{109}$$

(the $H^1(d, \ker\Delta)$ is generated by $x^m dx^m$). □

Case when P is a 5-form In this case we will write $P^{(5)}$ instead of P. Since $d\delta P^{(5)} = 0$, exists a harmonic 3-form $P^{(3)}$ such that:

$$\delta P^{(5)} = d P^{(3)} \tag{110}$$

Similarly, as $\delta d P^{(5)} = 0$, exists a harmonic 7-form $P^{(7)}$ such that:

$$d P^{(5)} = \delta P^{(7)} \tag{111}$$

Furthermore, there exist harmonic $P^{(1)}$ and $P^{(9)}$ such that:

$$\delta P^{(3)} = d P^{(1)} \texttt{ and } d P^{(7)} = \delta P^{(9)} \tag{112}$$

This implies that $\delta P^{(1)} = 0$ and therefore exists a harmonic form $S^{(2)}$ such that $P^{(1)} = \delta S^{(2)}$. Similarly, $P^{(9)} = dS^{(8)}$. Therefore the following P_L and P_R satisfy (108):

$$P_L = \frac{1}{2}\left(P^{(5)} - (P^{(3)} + dS^{(2)}) - (P^{(7)} + \delta S^{(8)})\right) \tag{113}$$

$$P_R = \frac{1}{2}\left(P^{(5)} + (P^{(3)} + dS^{(2)}) + (P^{(7)} + \delta S^{(8)})\right) \tag{114}$$

Case when P is a 3-form plus 7-form The 7-form part of P is related to the 3-form part by the condition that P is self-dual. In this case we will write $P^{(3)} + P^{(7)}$ instead of P. (This $P^{(3)}$ has nothing to do with the $P^{(3)}$ of the previous paragraph.) Since $d\delta P^{(3)} = 0$ and $\delta\delta P^{(3)} = 0$, exists harmonic $P^{(1)}$ such that:

$$\delta P^{(3)} = d P^{(1)} \tag{115}$$

This implies that $\delta P^{(1)} = 0$. Similarly, exists a harmonic $P^{(5)}$ such that:

$$dP^{(3)} = \delta P^{(5)} \tag{116}$$

This automatically implies:

$$dP^{(5)} = \delta P^{(7)} \tag{117}$$

Also exists a harmonic $P^{(9)}$ such that:

$$\delta P^{(9)} = dP^{(7)} \text{ and } dP^{(9)} = 0 \tag{118}$$

We take:

$$P_L = \frac{1}{2}\left(-P^{(1)} + P^{(3)} - P^{(5)} + P^{(7)} - P^{(9)}\right) \tag{119}$$

$$P_R = \frac{1}{2}\left(P^{(1)} + P^{(3)} + P^{(5)} + P^{(7)} + P^{(9)}\right) \tag{120}$$

Case when P is a 1-form plus a 9-form Now suppose that $P = P^{(1)} + P^{(9)}$. Let us first assume that the degree of P is more than 1. We have:

$$d\delta P^{(1)} = 0 \Rightarrow \delta P^{(1)} = 0 \Rightarrow P^{(1)} = \delta S^{(2)} \tag{121}$$

Similarly $P^{(9)} = dS^{(8)}$. Now we have:

$$P_L = \frac{1}{2}(\delta + d)S^{(2)} + \frac{1}{2}(d + \delta)S^{(8)} \tag{122}$$

$$P_R = \frac{1}{2}(\delta - d)S^{(2)} + \frac{1}{2}(d - \delta)S^{(8)} \tag{123}$$

Now consider the case when the degree of P is one, *i.e.* P is linear in x. In this case we can have $\delta P^{(1)} = \text{const}$. This corresponds to the R of (98) a constant proportional to unit matrix. The corresponding element of $H^1(Q_{\text{Lie}}\,,\ \text{Dirac} \otimes \text{Dirac})$ is:

$$(\theta_L \Gamma^m \lambda_L)\,(\theta_L \Gamma_m \hat{c} \Gamma_n \theta_R)\,(\lambda_R \Gamma^n \theta_R) \tag{124}$$

It corresponds to the following ghost number three vertex:

$$(\theta_L \Gamma^m \lambda_L)(\theta_L \Gamma^p \lambda_L)\,(\theta_L \Gamma_m \Gamma_p \Gamma_n \theta_R)\,(\lambda_R \Gamma^n \theta_R) \tag{125}$$

Conclusion We conclude that the main obstacle for (97) to be trivial is $\Gamma^m \partial_m R \neq 0$. (And besides that, there is also a case when R is a constant times a unit matrix, which results in a nontrivial vertex (125).) If $\Gamma^m \partial_m R \neq 0$, then there is a nontrivial cohomology class of the form:

$$c^n \Gamma_n R + r_1[c x_L x_R^{(2m-2)}] + r_2[c x_L^2 x_R^{(2m-3)}] + \ldots + r_{2m-1}[c x_L^{(2m-1)}] \tag{126}$$

Indeed, acting on the leading term $c^n \Gamma_n R$ with $-c^m \frac{\partial}{\partial x_R^m}$ we get an expression of the form $[c^2 x_R^{2m-2}]$, which does not depend on x_L and therefore is annihilated by $c^m \frac{\partial}{\partial x_L^m}$. But since $H^2\left(c \frac{\partial}{\partial x}\,,\ \text{Dirac}\right) = 0$, this expression is automatically of the form $c^m \frac{\partial}{\partial x_L^m}[c x_L x_R^{(2m-2)}]$. Continuing this process we get (126).

5.1.4. Proof that V_3 of Eq. (17) is BRST nontrivial

Let us consider the ghost number three vertex V_3 given by Eq. (17), and expand it in the Taylor series in x and θ. We assign to x degree 1 and to λ and θ degree $1/2$. The BRST operator preserves this degree. In particular, every term in the expansion is a BRST-closed *polynomial* of x, λ, θ. It is enough to prove the nontriviality term by term. Let us consider the extended space $(x_L, x_R, \lambda_L, \lambda_R, \theta_L, \theta_R)$. In this extended space, we get:

$$V_3 = (Q_L + Q_R)\,((a\cdot(x_L - x_R))V_2) \tag{127}$$

The corresponding element of $H^1\left(c\left(\frac{\partial}{\partial x_L} - \frac{\partial}{\partial x_R}\right),\ \text{Dirac}\otimes\text{Dirac}\right)$ is given by:

$$(a\cdot c)Pe^{k(x_L+x_R)} \tag{128}$$

Consider the expansion in powers of x_L. The leading term is $(a\cdot c)Pe^{kx_R}$. We observe:

$$\left(10(c\cdot a) - \hat{c}\hat{a}\right) = c\frac{\partial}{\partial x_L}(4\hat{x}_L\hat{a} + 5\hat{a}\hat{x}_L) \tag{129}$$

and $(4\hat{x}_L\hat{a} + 5\hat{a}\hat{x}_L)Pe^{kx_R}$ satisfies the left Dirac equation. Therefore (128) is equivalent to $\frac{1}{10}\hat{c}\hat{a}Pe^{kx_R}$. Comparing this with (126), we get:

$$R = \frac{1}{10}\hat{a}Pe^{kx_R} \tag{130}$$

$$\Gamma^m\frac{\partial}{\partial x_R^m}R = \frac{1}{5}(a\cdot k)Pe^{kx_R} \neq 0 \tag{131}$$

Then (102) implies that V_3 represents a nontrivial cohomology class.

Ghost number three vertex of [9] can be obtained as the first order of expansion of (128) in powers of x. Indeed, at the first order of the x-expansion $R = \frac{1}{10}\hat{a}(k\cdot x_R)P$. Notice that the expression:

$$\left(\hat{a}(k\cdot x_R) - \frac{1}{5}\hat{x}_R(k\cdot a)\right)P \tag{132}$$

satisfies the left Dirac equation (we use $\hat{k}P = 0$). Therefore $R = \frac{1}{10}\hat{a}(k\cdot x_R)P$ is equivalent to $R = \frac{1}{50}\hat{x}_R(k\cdot a)P$. Therefore the leading term of the x-linear part of (128) is equivalent to $\frac{1}{50}\hat{c}\hat{x}_R(k\cdot a)P$ which is the leading term of the vertex constructed in [9].

5.2. Maxwell–Maxwell sector

In this section we will compute the cohomology of $c^m\left(\frac{\partial}{\partial x_L^m} - \frac{\partial}{\partial x_R^m}\right)$ on the solutions of bi-Maxwell equations.

5.2.1. Bi-Maxwell equations

Solutions of bi-Maxwell equations are defined as expressions of the form:

$$dx_L^p \wedge dx_L^q\left(\frac{\partial}{\partial x_L^{[p}}\mathcal{A}(x_L, x_R)_{q]\,;\,[m}\frac{\overleftarrow{\partial}}{\partial x_R^{n]}}\right)dx_R^m \wedge dx_R^n \tag{133}$$

satisfying the left and right Maxwell equations:

$$\frac{\partial}{\partial x_L^p}\frac{\partial}{\partial x_L^{[p}}\mathcal{A}(x_L, x_R)_{q]\,;\,[m}\frac{\overleftarrow{\partial}}{\partial x_R^{n]}} = 0 \tag{134}$$

$$\frac{\partial}{\partial x_L^{[p}}\mathcal{A}(x_L, x_R)_{q]\,;\,[m}\frac{\overleftarrow{\partial}}{\partial x_R^{n]}}\frac{\overleftarrow{\partial}}{\partial x_R^{n}} = 0 \tag{135}$$

Notice that we have left and right indices, separated with the semicolon. We use the notations $\frac{\overleftarrow{\partial}}{\partial x}$. The expression $\phi\frac{\overleftarrow{\partial}}{\partial x}$ means the same as $\frac{\partial}{\partial x}\phi$. The sole purpose of such notations is to improve the readability of the formulas, as they allow us to naturally separate left and right indices.

5.2.2. *Spectral sequence $E_r^{p,q}$*

Definition As in Section 5.1, we will use the filtration by the powers of x_L, *i.e.* treat x_L as being small. The elements of $E_r^{p,q}$ are of the type:

$$E_r^{p,q} \;:\; dx_L \wedge dx_L\, [c^{p+q} x_L^{n+p} x_R^{n+q}]\, dx_R \wedge dx_R + \ldots \tag{136}$$

where $\ldots$ stands for terms of the type $dx_L \wedge dx_L\, [c^{p+q} x_L^{n+p+s} x_R^{n+q-s}]\, dx_R \wedge dx_R$ with $s > 0$, which are factored out when we consider F^p(Maxwell $\otimes$ Maxwell) modulo F^{p+1}(Maxwell $\otimes$ Maxwell). For a polynomial element $\mathcal{A}_{q\,;\,m}$, of the total order M in x_L and x_R, there is an expansion in powers of x_L:

$$\mathcal{A}(x_L, x_R) = \mathcal{A}^{(0)}_{q\,;\,m}(x_R) + \mathcal{A}^{(1)}_{q\,;\,m}(x_L, x_R) + \ldots + \mathcal{A}^{(N)}_{q\,;\,m}(x_L) \tag{137}$$

where $\mathcal{A}^{(0)}_{q\,;\,m}$ does not depend on x_L, $\mathcal{A}^{(1)}_{q\,;\,m}$ is linear in x_L, *etc.*

The structure of $E_2^{p,q}$ The following is the most general (up to the $c\frac{\partial}{\partial x_L}$-exact terms) ansatz for the leading term $\mathcal{A}^{(0)}_{q\,;\,m}$:

$$\begin{aligned}\mathcal{A}^{(0)}_{q\,;\,m} = \quad & c_p dx_L^p \wedge dx_L^q\, A(x_R)_{q\,;\,[m}\overleftarrow{\partial}_{n]}\; dx_R^m \wedge dx_R^n + \\ & + c^p dx_L^q \wedge dx_L^r\, B(x_R)_{pqr\,;\,[m}\overleftarrow{\partial}_{n]}\; dx_R^m \wedge dx_R^n\end{aligned} \tag{138}$$

$$\texttt{with } A(x_R)_{q\,;\,[m}\overleftarrow{\partial}_{n]}\overleftarrow{\partial}_m = 0 \tag{139}$$

$$B(x_R)_{pqr\,;\,[m}\overleftarrow{\partial}_{n]}\overleftarrow{\partial}_m = 0 \tag{140}$$

$$\partial^q A(x_R)_{q\,;\,[m}\overleftarrow{\partial}_{n]} = 0 \tag{141}$$

$$\partial_{[p} B(x_R)_{qrs]\,;\,[m}\overleftarrow{\partial}_{n]} = 0 \tag{142}$$

$$\mathcal{A}^{(0)}_{q\,;\,m}\ \texttt{represents an element of}\ E_2^{-\frac{M-1}{2},\frac{M+1}{2}}\ \texttt{, see (136)} \tag{143}$$

Here $A(x_R)_{q\,;\,m}$ and $B(x_R)_{pqr\,;\,m} = B(x_R)_{[pqr]\,;\,m}$ are polynomials in x_R of the order M. They correspond to the two terms in (57). Eqs. (139) and (140) enforce the right Maxwell equation. (The left Maxwell equation is automatically satisfied because A does not depend on x_L.) Eqs. (141) and (142) are the conditions for being in the kernel of d_1. In other words, those are the conditions for the existence of $A^{(1)}(c, x_L, x_R)_{p\,;\,m}$ linear in x_L and c such that:

$$c^r \frac{\partial}{\partial x_R^r}\Big(c_p dx_L^p \wedge dx_L^q \; A(x_R)_{q\,;\,[m} \overleftarrow{\partial}_{n]} \; dx_R^m \wedge dx_R^n + $$
$$+ \; c^p dx_L^q \wedge dx_L^r \; B(x_R)_{pqr\,;\,[m} \overleftarrow{\partial}_{n]} \; dx_R^m \wedge dx_R^n \Big) =$$
$$= c^r \frac{\partial}{\partial x_L^r} \Big(dx_L^p \wedge dx_L^q \; \partial_{[p} A^{(1)}(c, x_L, x_R)_{q]\,;\,[m} \overleftarrow{\partial}_{n]} \; dx_R^m \wedge dx_R^n \Big) \tag{144}$$

$$\text{and} \quad A^{(1)}(x_L, x_R)_{q\,;\,[m} \frac{\overleftarrow{\partial}}{\partial x_R^{n]}} \frac{\overleftarrow{\partial}}{\partial x_R^m} = \frac{\partial}{\partial x_L^p} \frac{\partial}{\partial x_L^{[p}} A^{(1)}(x_L, x_R)_{q]\,;\,m} = 0 \tag{145}$$

Eq. (141) is the vanishing of the obstacle proportional to the first term in (58), and Eq. (142) is to avoid hitting the second term in (58).

Remember that we are working in the polynomial sector, *i.e.* $B_{pqr\,;\,[m}(x_R) \overleftarrow{\partial}_{n]}$ is a homogeneous polynomial in x_R. Let us first assume that the degree of the polynomial is nonzero:

$$B_{pqr\,;\,[m}(x_R) \overleftarrow{\partial}_{n]} \neq \text{const} \tag{146}$$

Then (142) implies that we can remove the term $c^p dx_L^q \wedge dx_L^r \; B(x_R)_{pqr\,;\,[m} \overleftarrow{\partial}_{n]} \; dx_R^m \wedge dx_R^n$, by adding to $\mathcal{A}^{(0)}$ an element in $d_1(E_1^{-\frac{M+1}{2}, \frac{M+1}{2}})$. Indeed, this is equivalent to the existence of the following two objects:

- $C(x_R)_{pq\,;\,m}$ satisfying $C(x_R)_{pq\,;\,[m} \overleftarrow{\partial}_{n]} \overleftarrow{\partial}_m = 0$ and
- $G(x_L, x_R)_{pq\,;\,m}$ linear in x_L satisfying left and right Maxwell equations:

$$\frac{\partial}{\partial x_L^{[p}} G(x_L, x_R)_{qr]\,;\,[m} \frac{\overleftarrow{\partial}}{\partial x_R^{n]}} = 0$$
$$\frac{\partial}{\partial x_L^p} G(x_L, x_R)_{pq\,;\,[m} \frac{\overleftarrow{\partial}}{\partial x_R^{n]}} = 0$$
$$G(x_L, x_R)_{pq\,;\,[m} \frac{\overleftarrow{\partial}}{\partial x_R^{n]}} \frac{\overleftarrow{\partial}}{\partial x_R^m} = 0$$

such that:

$$c^p dx_L^q \wedge dx_L^r \; B(x_R)_{pqr\,;\,[m} \overleftarrow{\partial}_{n]} \; dx_R^m \wedge dx_R^n \; + O(x_L) =$$
$$= \; c^k \left(\frac{\partial}{\partial x_L^k} - \frac{\partial}{\partial x_R^k} \right) \Big(\; dx_L^p \wedge dx_L^q \; C(x_R)_{pq\,;\,[m} \overleftarrow{\partial}_{n]} \; dx_R^m \wedge dx_R^n +$$
$$+ \; dx_L^p \wedge dx_L^q \; G(x_L, x_R)_{pq\,;\,[m} \frac{\overleftarrow{\partial}}{\partial x_R^{n]}} dx_R^m \wedge dx_R^n \; \Big) +$$
$$+ \, c_p dx_L^p \wedge dx_L^q \; \widetilde{A}(x_R)_{q\,;\,[m} \overleftarrow{\partial}_{n]} \; dx_R^m \wedge dx_R^n \; + \; O(x_L) \tag{147}$$

Here $\widetilde{A}(x_R)_{q\,;\,[m} \overleftarrow{\partial}_{n]}$ is some correction to the $A(x_R)_{q\,;\,[m} \overleftarrow{\partial}_{n]}$ of (138). (In other words, when we gauge away the B-term, this leads to some change in the A term: $A \to A + \widetilde{A}$.) The existence of such $C(x_R)_{pq\,;\,m}$ and $G(x_L, x_R)_{pq\,;\,m}$ follows from (142) and the fact that $H^3(Q_{\text{Lie}}, \text{Maxw})$

is zero in polynomials of the degree > 0, in the following way.[6] Eq. (142) implies that exists $C(x_R)_{pq\,;\,m}$ satisfying the right Maxwell equation, such that:

$$B_{pqr\,;\,[m} \overleftarrow{\partial}_{n]} = -\partial_{[p} C_{qr]\,;\,[m} \overleftarrow{\partial}_{n]} \tag{148}$$

Therefore, in computing the first line on the RHS of (147), the $c^k \frac{\partial}{\partial x_L^k}$ gives zero as $C_{pq\,;\,m}$ does not depend on x_L, and when acting with $-c^k \frac{\partial}{\partial x_R^k}$, we get:

$$-c^k \frac{\partial}{\partial x_R^k}\, dx_L^p \wedge dx_L^q\, C(x_R)_{pq\,;\,[m} \overleftarrow{\partial}_{n]}\, dx_R^m \wedge dx_R^n \tag{149}$$

This has to be understood as an element of $E_1^{-\frac{M-1}{2},\frac{M+1}{2}}$, *i.e.* modulo the image of $c^k \frac{\partial}{\partial x_L^k}$. This ambiguity is described by the second line on the RHS of (147), the term containing $G(x_L, x_R)_{pq\,;\,m}$. This term can be used to remove the components other than those listed in Eq. (57); the component $\mathbf{C}^d$ corresponds to $\widetilde{A}_{q\,;\,m}$, and the component $\Lambda^3\mathbf{C}^d$ kills the B-term.

We conclude that we can get rid of the B-term in (138) by adding to $\mathcal{A}$ an element in $d_1(E_1^{-\frac{M+1}{2},\frac{M+1}{2}})$.

Now let us consider the case when $B_{pqr\,;\,[m}(x_R) \overleftarrow{\partial}_{n]}$ is constant:

$$B_{pqr\,;\,[m}(x_R) \overleftarrow{\partial}_{n]} = \text{const} \tag{150}$$

Consider the total antisymmetrization:

$$\mathcal{B}_{[pqrmn]} = B_{[pqr\,;\,m}(x_R) \overleftarrow{\partial}_{n]} \tag{151}$$

In this case the B-term in (138) cannot be gauged away, as $\mathcal{B}_{[pqrmn]}$ represents a nontrivial cohomology class of $H^3(\text{Maxw}) = \Lambda^5\mathbf{C}^{10}$. However, we will show in Section 7.5 that this is cancelled by the $d_2 : \mathcal{E}_2^{1,2} \to \mathcal{E}_2^{3,1}$. In other words, for our ansatz to survive on $\mathcal{E}_3^{1,2}$ we need to put $\mathcal{B}_{[pqrmn]}$ to zero:

$$\mathcal{B}_{[pqrmn]} = 0 \tag{152}$$

5.2.3. *Double field strength*

Let us therefore assume that $B(x_R)_{pqr\,;\,m} = 0$. Can the remaining A-term also be in the image of d_1? Let us define the double field strength $F_{[pq];[mn]}$ as follows:

$$F_{[pq];[mn]} = \partial_{[p} A_{q]\,;\,[m} \overleftarrow{\partial}_{n]} \tag{153}$$

This double field strength has the following properties:

$$F_{[pq\,;\,mn]} = 0 \ \texttt{(total antisymmetrization)} \tag{154}$$

$$\partial_{[p} F_{qr]\,;\,mn} = 0 \tag{155}$$

$$F_{qr\,;\,[mn} \overleftarrow{\partial}_{k]} = 0 \tag{156}$$

[6] Notice that we are using the results about $H(Q_{\text{Lie}}, \text{Maxw})$ in two different ways. First, we use $H^1(Q_{\text{Lie}}, \text{Maxw}_L)$ to argue that the leading term can be reduced to the form (138). Then we use the *vanishing* of $H^3(Q_{\text{Lie}}, \text{Maxw}_R)$ in polynomials of high enough degree to remove the B-term by adding d_1(smth).

$$\partial^p F_{[pq];[mn]} = 0 \tag{157}$$

$$F_{[pq];[mn]} \overleftarrow{\partial}^m = 0 \tag{158}$$

$$\Delta F_{[pq];[mn]} = 0 \tag{159}$$

5.2.4. *Double field strength is the obstacle to triviality*

We will now show that $\mathcal{A}$ is trivial iff $F_{pq\,;\,mn} = 0$.

We have to understand under which conditions the class with the leading term (138) is trivial, *i.e.* can be obtained by acting with $c^m \left(\frac{\partial}{\partial x_L^m} - \frac{\partial}{\partial x_R^m} \right)$ on something:

$$c_{[p} dx_L^p \wedge dx_L^q \; A(x_R)_{q]\,;\,[m} \overleftarrow{\partial}_{n]} \; dx_R^m \wedge dx_R^n \; + \ldots = \tag{160}$$
$$= c^j \left(\frac{\partial}{\partial x_L^j} - \frac{\partial}{\partial x_R^j} \right) \left(dx_L^p \wedge dx_L^q \; W(x_R)_{pq\,;\,[m} \frac{\overleftarrow{\partial}}{\partial x_R^{n]}} \; dx_R^m \wedge dx_R^n \; + \ldots \right)$$

This property is equivalent to the existence of $W(x_R)_{pq\,;\,m}$ satisfying:

$$A_{q\,;\,[m} \overleftarrow{\partial}_{n]} = \partial^p W_{pq\,;\,[m} \overleftarrow{\partial}_{n]} \tag{161}$$

$$\texttt{with} \quad \partial_{[p} W_{qr]\,;\,[m} \overleftarrow{\partial}_{n]} = 0 \tag{162}$$

$$\texttt{and} \quad W_{pq\,;\,[m} \overleftarrow{\partial}_{n]} \overleftarrow{\partial}_n = 0 \tag{163}$$

Notice that the ghost number two vertices correspond to $W_{pq\,;\,m}$ satisfying (162), (163) and $\partial^p W_{pq\,;\,[m} \overleftarrow{\partial}_{n]} = 0$ instead of (161).

If A can be expressed through W as in (161), then we have:

$$F_{pq\,;\,mn} = \partial_{[p} A_{q]\,;\,[m} \overleftarrow{\partial}_{n]} = -\,\partial_{[p} \partial^r W_{q]r\,;\,[m} \overleftarrow{\partial}_{n]} = \frac{1}{2} \partial_r \partial^r W_{pq\,;\,[m} \overleftarrow{\partial}_{n]} = 0 \tag{164}$$

This means that:

$$F_{pq\,;\,mn} \neq 0 \; \texttt{is an obstacle to the triviality of } \mathcal{A} \tag{165}$$

We will now prove that this is the only obstacle. In other words, if $F_{pq\,;\,mn} = 0$, then (137) is cohomologically trivial.

Let $\mathcal{M}$ be the space of polynomial expressions of the form:

$$\Phi(dx_L, x)_{[mn]} \; \texttt{satisfying} \;\; \Phi_{[mn} \overleftarrow{\partial}_{k]} = 0 \; \texttt{and} \;\; \Phi_{mn} \overleftarrow{\partial}_n = 0 \tag{166}$$

Let $\mathcal{M}^N$ be the subspace of $\mathcal{M}$ consisting of polynomials of the order N in x, *i.e.* $x^p \frac{\partial}{\partial x^p} \Phi = N\Phi$. Notice that such Φ_{mn} are automatically harmonic. The operator $d_L + \delta_L$ acts on such expressions, and is nilpotent:

$$\ldots \longrightarrow \mathcal{M}^N \xrightarrow{d_L + \delta_L} \mathcal{M}^{N-1} \longrightarrow \ldots \tag{167}$$

Lemma

$$H^N(d_L + \delta_L \,,\, \mathcal{M}) = H^N(d_L \,,\, \mathcal{M}) = H^N(\delta_L \,,\, \mathcal{M}) = 0 \; \texttt{for} \;\; N > 0 \tag{168}$$

Indeed, d_L is acyclic, as $H^N(d_L)$ is the same as already computed in Section 3.1 cohomology of the translations algebra on the solutions of the Maxwell equations, and it is zero for $N > 0$.

This means that it is always possible to gauge away the term with the highest number of dx_L, and therefore the cohomology of $d_L + \delta_L$ is zero. The proof of $H^N(\delta_L) = 0$ is identical to the proof of $H^N(d_L) = 0$ after applying the Hodge dual operation on the c ghosts.

Eq. (139) implies that the expression $dx_L^p\, A_{p\,;\,[m} \overleftarrow{\partial}_{n]}$ belongs to $\mathcal{M}$. Eq. (141) implies that it is annihilated by δ_L. Since $H^N(\delta_L) = 0$, exists $\Phi^{(2)} \in \mathcal{M}$ such that:

$$dx_L^p\, A_{p\,;\,[m} \overleftarrow{\partial}_{n]} = \delta_L \left(dx_L^p \wedge dx_L^q\, \Phi^{(2)}_{pq\,;\,mn} \right) \tag{169}$$

Now suppose that $F_{pq\,;\,mn} = 0$. This implies that we can find $\Phi^{(4)}$, $\Phi^{(6)}$, $\Phi^{(8)}$ and $\Phi^{(10)}$ (all elements of $\mathcal{M}$) satisfying:

$$dx_L^p\, A_{p\,;\,[m} \overleftarrow{\partial}_{n]} = (\delta_L + d_L) \left(\Phi^{(2)}_{mn} + \Phi^{(4)}_{mn} + \Phi^{(6)}_{mn} + \Phi^{(8)}_{mn} + \Phi^{(10)}_{mn} \right) \tag{170}$$

(Here each $\Phi^{(2j)}_{mn}$ is a polynomial of the degree $2j$ in dx_L.) Indeed, as elements of $\mathcal{M}$ are harmonic functions, $d_L \delta_L \Phi^{(2)}_{mn} = 0$ implies $\delta_L d_L \Phi^{(2)}_{mn} = 0$ and therefore the existence of $\Phi^{(4)}$ such that $d_L \Phi^{(2)}_{mn} + \delta_L \Phi^{(4)}_{mn} = 0$. And so on until $\Phi^{(10)}_{mn}$.

Since $\Phi^{(10)}_{mn}$ is a top form, exists $\Psi^{(9)} \in \mathcal{M}$ such that $\Phi^{10} = d_L \Psi^{(9)}$. Furthermore, $d_L(\Phi^8 - \delta_L \Psi^{(9)}) = 0$, therefore exists $\Psi^{(7)} \in \mathcal{M}$ such that $\Phi^8 - \delta_L \Psi^{(9)} = d_L \Psi^{(7)}$. Continuing, we get $\Phi^{(6)} - \delta_L \Psi^{(7)} = d_L \Psi^{(5)}$, $\Phi^{(4)} - \delta_L \Psi^{(5)} = d_L \Psi^{(3)}$ and finally $d_L(\Phi^{(2)} - \delta_L \Psi^{(3)}) = 0$. Let us denote:

$$\Phi = \Phi^{(2)} + \Phi^{(4)} + \Phi^{(6)} + \Phi^{(8)} + \Phi^{(10)} \tag{171}$$

$$\Psi = \Psi^{(3)} + \Psi^{(5)} + \Psi^{(7)} + \Psi^{(9)} \tag{172}$$

Then we get:

$$dx_L^p\, A_{p\,;\,[m} \overleftarrow{\partial}_{n]} = (\delta_L + d_L) \Big(\Phi - (\delta_L + d_L)\Psi \Big) \tag{173}$$

Notice that $\Phi - (\delta_L + d_L)\Psi$ is of the form:

$$\Phi - (\delta_L + d_L)\Psi = dx_L^p \wedge dx_L^q\, \widetilde{\Psi}_{pq\,;\,mn} \tag{174}$$

This concludes the proof that the ansatz (138) is trivial iff $F_{[pq]\,;\,[mn]} = 0$.

Case $N = 0$ The vanishing lemma (168) does not work in the case $N = 0$, in this case the cohomology of d_L is given by the formulas of Section 3.1 with the replacement $c^m \mapsto dx_L^m$, $dx^m \mapsto dx_R^m$. Similarly, the cohomology of δ_L is obtained via the Hodge duality. Therefore, it is necessary to repeat the analysis taking into account this nontrivial cohomology. There is no obstacle to satisfy (169), even if $A_{p\,;\,[m} \overleftarrow{\partial}_{n]} =$ const, because there are no 11-forms and therefore the cohomology of δ_L vanishes on expressions which are monomials of the first order in dx_L. There are potential obstacles in completing the chain (170). We will not do the analysis here, but just point out that by rotational symmetry, the potential obstacles are proportional to the following constant tensors: the total antisymmetrization and the contraction:

$$\mathcal{C}_{pmn} = A_{[p\,;\,m} \overleftarrow{\partial}_{n]} \tag{175}$$

$$\mathcal{C}_n = g^{pm} A_{p\,;\,[m} \overleftarrow{\partial}_{n]} \tag{176}$$

5.3. Dirac–Maxwell sector

Consider the following ansatz for the leading term of the expansion in powers of x_L:

$$\hat{c}\Psi_{[m}(x_R)\overleftarrow{\partial}_{n]}\, dx_R^m \wedge dx_R^n \tag{177}$$

where Ψ satisfies the Maxwell equation $\Psi\overleftarrow{\mathbf{M}} = 0$. This is in the image of d_1 when exists Φ such that:

$$\Psi_{[m}\overleftarrow{\partial}_{n]} = \Gamma^k \partial_k \Phi_{[m}\overleftarrow{\partial}_{n]} \tag{178}$$

$$\texttt{and} \quad \Phi\overleftarrow{\mathbf{M}} = 0$$

Then it follows that $\Delta\Phi_{[\bullet}\overleftarrow{\partial}_{\bullet]} = 0$ and therefore:

$$\Gamma^m \partial_m \Psi_{[\bullet}\overleftarrow{\partial}_{\bullet]} = 0 \tag{179}$$

If Ψ does not satisfy this equation, then the trivialization (178) is impossible. Notice that $\Psi\overleftarrow{\mathbf{M}} = 0$, therefore $\Psi_{[\bullet}\overleftarrow{\partial}_{\bullet]}$ is automatically annihilated by Δ. But it is not necessarily annihilated by the left Dirac operator.

We conclude that $\Gamma^m \partial_m \Psi_{[\bullet}\overleftarrow{\partial}_{\bullet]}$ is an obstacle for (177) to be trivial.

5.4. Maxwell–Dirac sector

Consider the following ansatz for the leading term:

$$\Psi_m(x_R)\, c_n dx_L^n \wedge dx_L^m \tag{180}$$

where Ψ satisfies the right Dirac equation $\Psi_m \overleftarrow{\partial}_n \Gamma^n = 0$ and also $\partial^m \Psi_m = 0$. This is trivial if exists A_m such that:

$$\Psi_m = \partial_n(\partial_n A_m - \partial_m A_n) \tag{181}$$

$$\texttt{with} \quad A\overleftarrow{\partial}_k \Gamma^k = 0 \tag{182}$$

This implies that $\Delta A = 0$ and therefore $\partial_{[\bullet}\Psi_{\bullet]} = 0$. Therefore $\partial_{[\bullet}\Psi_{\bullet]}$ is an obstacle for (180) to be trivial. For the polynomials of nonzero degree this is the only obstacle. Indeed, suppose that $\partial_{[m}\Psi_{n]} = 0$. As the cohomology of Dirac solutions at the nonzero degree is zero, this implies that:

$$\Psi_m = \partial_m \Xi \tag{183}$$

where $\Xi = \Xi(x_R)$ satisfies the right Dirac equation. The cohomology of $H^{<9}(\delta)$ on the solutions of the Dirac equation is zero, therefore exists A_n such that $\Xi = -\partial^n A_n$ where Φ_n satisfies the Dirac equation. This implies (181).

6. Second cohomology of the tensor product of two classical electrodynamics

The term $E_2^{1-\frac{M}{2},\,1+\frac{M}{2}}$ is generated by two types of terms:

$$c_p c_q dx_L^p \wedge dx_L^q \; A_{[m}(x_R) \overleftarrow{\partial}_{n]} \; dx_R^m \wedge dx_R^n +$$
$$+ c^p c^q dx_L^r \wedge dx_L^s \; B_{[pqrs]\,;\,[m}(x_R) \overleftarrow{\partial}_{n]} \; dx_R^m \wedge dx_R^n \tag{184}$$

where $A_{[m}(x_R) \overleftarrow{\partial}_{n]}$ is a polynomial of degree M in x_R. Under $d_2 : E_2^{-1-\frac{M}{2}, 2+\frac{M}{2}} \to E_2^{1-\frac{M}{2}, 1+\frac{M}{2}}$ the first term cancels with the right hand side of Eq. (141), because $H^9(\text{Maxw}) = 0$. The second term for nonconstant $B_{[pqrs]\,;\,[m} \overleftarrow{\partial}_{n]}$ cancels with the right hand side of Eq. (142). The constant $B_{[pqrs]\,;\,[m} \overleftarrow{\partial}_{n]} = \text{const}$ (*i.e.* $M = 0$) generates $\Lambda^6 \mathbf{C}^{10}$ (because $H^4(\text{Maxw}) = \Lambda^6 \mathbf{C}^{10}$; the d_2 acts as Q_{Lie} on Maxwell solutions):

$$H^2(\text{SM}_L \otimes \text{SM}_R) = \Lambda^6 \mathbf{C}^{10} \tag{185}$$

7. BRST cohomology

We are now ready to compute the cohomology of Q_{SUGRA}.

7.1. Ghost number one

The corresponding part of $\mathcal{E}_2$ consists of two parts:

$$\mathcal{E}_2^{1,0} = \mathbf{C}^{10} \tag{186}$$
$$\mathcal{E}_2^{0,1} = H^0(\text{SMaxw}_L) \bigoplus H^0(\text{SMaxw}_R) =$$
$$= \Lambda^2 \mathbf{C}^{10} \bigoplus \Lambda^2 \mathbf{C}^{10} \bigoplus \mathbf{C}^{16} \bigoplus \mathbf{C}^{16} \tag{187}$$

However, there is a nontrivial $d_2 : \mathcal{E}_2^{0,1} \to \mathcal{E}_2^{2,0} = \Lambda^2 \mathbf{C}^{10}$, which cancels the $L \leftrightarrow R$ antisymmetric part of $\Lambda^2 \mathbf{C}^{10} \bigoplus \Lambda^2 \mathbf{C}^{10} \subset \mathcal{E}_2^{0,1}$ with $\mathcal{E}_2^{2,0}$. We are left with:

$$\mathcal{E}_\infty^{1,0} = \mathbf{C}^{10} \tag{188}$$
$$\mathcal{E}_\infty^{0,1} = \Lambda^2 \mathbf{C}^{10} \bigoplus \mathbf{C}^{16} \bigoplus \mathbf{C}^{16} \tag{189}$$
$$\mathcal{E}_\infty^{2,0} = 0 \tag{190}$$

These vertices are in one-to-one correspondence with the generators of the super-Poincare algebra.

7.2. Ghost number two

The corresponding part of $\mathcal{E}_2$ consists of three parts:

$$\mathcal{E}_2^{2,0} = \Lambda^2 \mathbf{C}^{10} \tag{191}$$
$$\mathcal{E}_2^{1,1} = H^1(\text{SMaxw}_L) \bigoplus H^1(\text{SMaxw}_R) \tag{192}$$
$$\mathcal{E}_2^{0,2} = H^0(\text{SMaxw}_L \otimes \text{SMaxw}_R) \tag{193}$$

7.2.1. $\mathcal{E}_2^{2,0}$

We have already seen that $\mathcal{E}_2^{2,0}$ gets killed by the d_2:

$$\mathcal{E}_\infty^{2,0} = 0 \tag{194}$$

7.2.2. $\mathcal{E}_2^{1,1}$

Let us look at $\mathcal{E}_2^{1,1}$. We have:

$$\mathcal{E}_2^{1,1} = \left(\mathbf{C}^{10} \oplus \Lambda^3\mathbf{C}^{10} \oplus \mathbf{C}^{16}\right) \bigoplus \left(\mathbf{C}^{10} \oplus \Lambda^3\mathbf{C}^{10} \oplus \mathbf{C}^{16}\right) \tag{195}$$

The interpretation is as follows:

- $\mathbf{C}^{10} \oplus \mathbf{C}^{10}$ corresponds to the linear dilaton and the "asymmetric linear dilaton" (the non-physical vertex of [3] with constant A_m^-)
- One copy of $\Lambda^3\mathbf{C}^{10}$ cancels under d_2 with $\mathcal{E}_2^{3,0}$
- Another copy of $\Lambda^3\mathbf{C}^{10}$ is the NSNS B-field strength $H = dB$
- Two copies of $\mathbf{C}^{16}$ are both unphysical

7.2.3. $\mathcal{E}_2^{0,2}$

This was computed in Section 4. We identify A_m^+ as $\partial_m\Phi$ (the gradient of the dilaton) and A_m^- is the unphysical state of [3]. Notice that Eq. (86) implies that $\partial_p\partial_q A_m^- = 0$, *i.e.* A_m^- is a linear function of x. Notice that Eqs. (75) and (74) define $\phi^{L|R}$ only up to a constant, and therefore $A_m^\pm$ is defined only up to a constant. This is because linear dilaton and linear asymmetric dilaton have already been counted in $\mathcal{E}_\infty^{1,1}$.

Conclusion As expected, $F_{pq\,;\,mn}$ has the quantum numbers of the NSNS sector of the linearized Type IIB SUGRA, modulo some zero mode subtleties. The symmetric part $F_{pq\,;\,mn} + F_{mn\,;\,pq}$ corresponds to the Riemann curvature tensor $R_{[pq][mn]}$, and the antisymmetric part $F_{pq\,;\,mn} - F_{mn\,;\,pq}$ to $\partial_{[p}B^{\mathrm{NSNS}}_{q]\,[m}\overleftarrow{\partial}_{n]}$.

7.3. Comment on nonphysical states

There are the following nonphysical states:

$\mathbf{C}^{10}$	from $\mathcal{E}_\infty^{1,1}$: constant A_m^-
$\mathbf{C}^{16} \oplus \mathbf{C}^{16}$	from $\mathcal{E}_\infty^{1,1}$
$\Lambda^2\mathbf{C}^{10}$	from $\mathcal{E}_\infty^{0,2}$: $\partial_{[q}A_{m]}^-$

They have the quantum numbers of the adjoint representation of the super-Poincare algebra.

In the bosonic string theory, the nonphysical states were removed by imposing the constraint $(b_0 - \overline{b}_0)V = 0$ [2]. This is probably possible also in the pure spinor approach, as the pure spinor b-ghost was constructed in the nonminimal formalism [11]. But there is also another way of removing the nonphysical states, which we will now describe.

As we discussed in the Introduction, the BRST closedness of the vertex operator is a necessary and sufficient condition for the corresponding deformation of the classical worldsheet action to have the classical BRST invariance. However, at the one-loop level there is an anomaly which is cancelled by the Fradkin–Tseytlin term [6]:

$$\alpha' \int d^2\tau\ \Phi R \tag{196}$$

Here Φ is the *dilaton superfield*. The only place where Φ enters is the Fradkin–Tseytlin term (196), which does not matter at the *classical* level. It is, in this sense, "invisible" in the classical theory. The condition of the one-loop BRST invariance implies that Φ is related to the "visible" superfields (those which enter in the main part of the worldsheet action) by some equations:

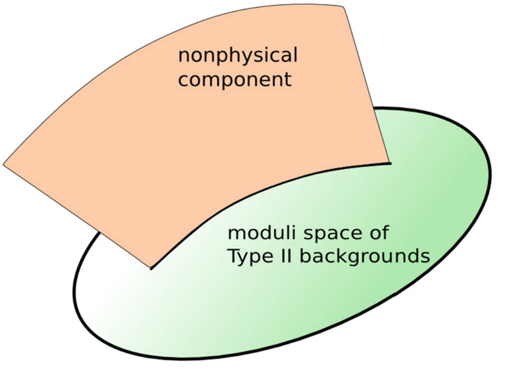
$$D_\alpha \Phi = \Omega_\alpha \tag{197}$$

$$D_{\hat\alpha} \Phi = \widehat{\Omega}_{\hat\alpha} \tag{198}$$

where Ω_α and $\widehat{\Omega}_{\hat\alpha}$ on the right hand side are some function of the "visible" superfields. In this sense, Φ is determined, unambiguously up to a constant, from the "visible" superfields.

However, it turns out that for some classical backgrounds the equations (197) and (198) are incompatible [3]. Such backgrounds, in our terminology, are *nonphysical*. Being perfectly consistent from the point of view of the classical worldsheet sigma-model, they however fail at the one-loop level.

This is somewhat unusual, as the typical situation is that differential equations are "generally speaking incompatible, but sometimes become compatible". Here we have the opposite situation. Equations (197) and (198) for Φ are compatible for the vast majority of backgrounds, but become incompatible on a finite-dimensional nonphysical component. In other words, **physical and nonphysical deformations are "mutually obstructed"**.



Roughly speaking, this can be understood as follows. The compatibility conditions for equations (197) and (198) include the equation:

$$\Gamma_m^{\alpha\beta} D_\alpha \Omega_\beta - \Gamma_m^{\hat\alpha\hat\beta} D_{\hat\alpha} \Omega_{\hat\beta} = 0 \tag{199}$$

Both Ω_α and $\hat\Omega_{\hat\alpha}$ are defined in terms of other SUGRA fields, which already satisfy the SUGRA constraints. These constraints translate into some constraint on $\Gamma_m^{\alpha\beta} D_\alpha \Omega_\beta - \Gamma_m^{\hat\alpha\hat\beta} D_{\hat\alpha} \Omega_{\hat\beta}$ (which is therefore automatically satisfied). Surprisingly, that automatic constraint seems to be not $\Gamma_m^{\alpha\beta} D_\alpha \Omega_\beta - \Gamma_m^{\hat\alpha\hat\beta} D_{\hat\alpha} \Omega_{\hat\beta} = 0$ but rather $\Gamma_m^{\alpha\beta} D_\alpha \Omega_\beta - \Gamma_m^{\hat\alpha\hat\beta} D_{\hat\alpha} \Omega_{\hat\beta} = \text{const}$, *i.e.* the *derivatives* of $\Gamma_m^{\alpha\beta} D_\alpha \Omega_\beta - \Gamma_m^{\hat\alpha\hat\beta} D_{\hat\alpha} \Omega_{\hat\beta}$ being zero ([9,3], *cp.* Eqs. (71) and (72)). In order to kill the nonphysical component, we just have to require that this constant is zero; this is why the nonphysical component is finite-dimensional.

We observe that the nonphysical operators seem to be in correspondence with the global symmetries. This should have a natural interpretation in terms of the action of the b-ghost:

$$\boxed{\text{nonphysical, ghost number 2}} \xrightarrow{b_0-\bar{b}_0} \boxed{\begin{array}{c}\text{ghost number 1}\\ \text{(global symmetries)}\end{array}} \tag{200}$$

But, as we explained:

- instead of imposing the condition $(b_0 - \overline{b}_0)V = 0$, one can request the existence of the dilaton superfield Φ

Notice that including Φ also solves the following problem. Our analysis, based on the naive BRST cohomology, failed to identify the dilaton zero mode. But once we include Φ, the dilaton is identified as the lowest component of Φ, and in particular the zero mode of the dilaton is recovered.

7.4. Ghost number three

Most of the ghost number three vertex operators transform in the same representation as ghost number two vertex operators. This is in line with the picture:

$$\boxed{\text{ghost number 3}} \xrightarrow{b_0 - \overline{b}_0} \boxed{\text{ghost number 2}} \tag{201}$$

Notice that the map (201) lowers the polynomial degree of the vertex by 2, as the b-ghost should. For example, in the Dirac–Dirac sector, the ghost number 3 vertex is of the form $\hat{c}R$; to produce the bispinor field we remove $\hat{c}$ and then act with the left Dirac operator:

$$\hat{c}R \mapsto \Gamma^m \frac{\partial}{\partial x_R^m} R \tag{202}$$

Removing $\hat{c}$ lowers the degree by one, and then $\frac{\partial}{\partial x_R^m}$ again lowers the degree by one.

Let us look more carefully at the subtleties which arise when we consider polynomial vertices of low degree.

7.4.1. $\mathcal{E}_2^{3,0}$

This is $\Lambda^3\mathbf{C}^{10}$. It cancels with part of $\mathcal{E}_2^{1,1}$ — see Section 7.2.2.

7.4.2. $\mathcal{E}_2^{2,1}$

This is $\Lambda^4\mathbf{C}^{10} \oplus \mathbf{C} \oplus \Lambda^4\mathbf{C}^{10} \oplus \mathbf{C}$. First of all, we have restrict to the kernel of $d_2 : \mathcal{E}_2^{2,1} \to \mathcal{E}_2^{4,0}$. This kills one copy of $\Lambda^4\mathbf{C}^{10}$. But also, we have to take a factorspace over the image of $d_2 : \mathcal{E}^{0,2} \to \mathcal{E}^{2,1}$. This cancels another copy of $\Lambda^4\mathbf{C}^{10}$ against the (71) and one copy of $\mathbf{C}$ against the (72). For example, $d_2(dx_L^m x_L^n f_{mnpq} dx_R^p x_R^q)$ cancels the diagonal $\Lambda^4\mathbf{C}^{10}$ as:

$$\begin{aligned}
& dx_L^m x_L^n f_{mnpq} dx_R^p x_R^q \xrightarrow{Q_{\text{Lie}}} \\
\xrightarrow{Q_{\text{Lie}}} \; & - dx_L^m c^n f_{mnpq} dx_R^p x_R^q - dx_L^m x_L^n f_{mnpq} dx_R^p c^q \xrightarrow{(Q_L+Q_R)^{-1}} \\
\xrightarrow{(Q_L+Q_R)^{-1}} \; & - x_L^m c^n f_{mnpq} dx_R^p x_R^q + dx_L^m x_L^n f_{mnpq} x_R^p c^q \xrightarrow{Q_{\text{Lie}}} \\
\xrightarrow{Q_{\text{Lie}}} \; & - c^m c^n f_{mnpq} dx_R^p x_R^q - dx_L^m x_L^n f_{mnpq} c^p c^q
\end{aligned} \tag{203}$$

and a similar computation shows that $d_2((dx_L \cdot x_R)(dx_R \cdot x_L))$ cancels a diagonal copy of $\mathbf{C}$. Another copy of $\mathbf{C}$ does not seem to cancel with anything:

$$\mathcal{E}_\infty^{2,1} = \mathbf{C} \tag{204}$$

7.4.3. $\mathcal{E}_2^{1,2}$

Dirac–Dirac sector There are the following obstacles to triviality:

1. The bispinor $\Gamma^m \frac{\partial}{\partial x_R^m} R$, which satisfies both left and right Dirac equations
2. There is also a discrete state (125) which corresponds to R being a constant times the unit matrix

Maxwell–Maxwell sector There are the following obstacles to triviality:

1. The double field strength $F_{[mn]\,;\,[pq]}$ of Section 5.2.3
2. If the double field strength is zero, then there are constant tensors $\mathcal{C}_{klm}$ and $\mathcal{C}_k$ defined in (175) and (176)

First let us look at the double field strength. Notice that the formulas of Section 5.2.3 are almost identical to Section (7.2.3 → 4). The only difference is that the $F_{[mn]\,;\,[pq]}$ of Section 4 is not required to be of the form (153) with $A_{n\,;\,p}$ satisfying Eq. (141). As the $F_{[mn]\,;\,[pq]}$ of Section 5.2.3 *is* required to be of such a form, it automatically satisfies:

$$g^{np} F_{[mn]\,;\,[pq]}(x) = \frac{1}{2} \partial_{[m} g^{np} A_{n]\,;\,[p} \overleftarrow{\partial}_{q]} \tag{205}$$

Eq. (141) implies the existence of Φ such that $\partial^n A_{n\,;\,p} = \partial_p \Phi$. Taking also into account Eq. (139), we get:

$$g^{np} F_{[mn]\,;\,[pq]}(x) = \partial_m \partial_q \left[\frac{1}{8} \left(g^{pn} A_{p\,;\,n} - \Phi \right) \right] \tag{206}$$

This is the same equation as we got in Section 4, except there is no unphysical A_m^-.

On the other hand, there are $\mathcal{C}_{klm}$ and $\mathcal{C}_k$ defined in (175) and (176), which should be mapped by $b_0 - \overline{b}_0$ to H_{klm}^{NSNS} and the dilaton gradient. Also, there is the discrete state (125).

Notice that in our computation we missed the dilaton zero mode, as the corresponding vertex is probably a BRST variation of something that is not annihilated by $b_0 - \bar{b}_0$ [2]. It is possible that the discrete state (125) mapped by $b_0 - \bar{b}_0$ to the dilaton zero mode. However, there is also another discrete state at the ghost number three: Eq. (204). Therefore our computations seem to confirm Eq. (201), except that we see *two* ghost scalar ghost number three discrete states: Eq. (125) and Eq. (204).

7.5. Ghost number four

7.5.1. $\mathcal{E}_\infty^{2,2}$

The term $\mathcal{E}_2^{2,2} = H^2(Q_{\text{Lie}}\,,\ \text{SMaxw}_L \otimes \text{SMaxw}_R)$ was computed in Section 6:

$$H^2(\text{SM}_L \otimes \text{SM}_R) = \Lambda^6 \mathbf{C}^{10} \tag{207}$$

It cancels with half of:

$$\mathcal{E}_2^{4,1} = H^4(\text{SM}_L \oplus \text{SM}_R) = \Lambda^6 \mathbf{C}^{10} \oplus \Lambda^6 \mathbf{C}^{10} \tag{208}$$

(and another half of $\mathcal{E}_2^{4,1}$ then cancels with $\mathcal{E}_2^{5,0} = \Lambda^6 \mathbf{C}^{10}$). This pattern persists for $2 < p \leq 6$, giving the short exact sequences:

$$0 \longrightarrow [\mathcal{E}^{p,2} = \Lambda^{p+4}\mathbf{C}^{10}] \xrightarrow{d_2}$$
$$\xrightarrow{d_2} [\mathcal{E}^{p+2,1} = \Lambda^{p+4}\mathbf{C}^{10} \oplus \Lambda^{p+4}\mathbf{C}^{10}] \xrightarrow{d_2}$$
$$\xrightarrow{d_2} [\mathcal{E}^{p+4,0} = \Lambda^{p+4}\mathbf{C}^{10}] \tag{209}$$

7.5.2. $\mathcal{E}_\infty^{4,0}$

The term $H^4(Q_{\text{Lie}}\ ,\ \mathbf{C}) = \Lambda^4\mathbf{C}^{10}$ is nonzero, but it cancels with the d_2 of $H^2(Q_{\text{Lie}}\ ,\ \text{SMaxw}_L \bigoplus \text{SMaxw}_R)$.

7.5.3. $\mathcal{E}_\infty^{3,1}$

The space $\ker\Big(d_2 :\ H^3(Q_{\text{Lie}}\ ,\ \text{SMaxw}_L \bigoplus \text{SMaxw}_R) \to H^5(Q_{\text{Lie}}\ ,\ \mathbf{C})\Big)$ is killed by the d_2 of $H^1(Q_{\text{Lie}}\ ,\ \text{SMaxw}_L \otimes \text{SMaxw}_R)$. Indeed, let us consider the following element of $H^1(Q_{\text{Lie}}\ ,\ \text{SMaxw}_L \otimes \text{SMaxw}_R)$ with constant $B_{pqr\ ;\ [m\ \overleftarrow{\partial}\ n]}$:

$$c^p dx_L^q \wedge dx_L^r\ B_{pqr\ ;\ [m\ \overleftarrow{\partial}\ n]}\ dx_R^m \wedge dx_R^n \tag{210}$$

(This is a particular case of (138) with zero A and constant B.) Being an element of $H^1(Q_{\text{Lie}}\ ,\ \text{SMaxw}_L \otimes \text{SMaxw}_R)$, this is a c-dependent element of the cohomology of $Q_L + Q_R$, parametrized by a left times right field strength. We need to act on this by the $d_2\ :\ \mathcal{E}_2^{1,2} \to \mathcal{E}_2^{3,1}$. For that, we need to know the actual (c-dependent) vertex, which is built using the left and right *vector potentials*, *i.e.* $c^p\ x_L^q(\theta_L\Gamma^r\lambda_L)\ B_{pqr\ ;\ [m\ \overleftarrow{\partial}\ n]}\ x_R^m(\theta_R\Gamma^n\lambda_R)$. The Q_{Lie} on the vertex is not zero:

$$-c^p c^q(\theta_L\Gamma^r\lambda_L)B_{pqr\ ;\ [m\ \overleftarrow{\partial}\ n]}\,x_R^m(\theta_R\Gamma^n\lambda_R) - c^p x_L^q(\theta_L\Gamma^r\lambda_L)B_{pqr\ ;\ [m\ \overleftarrow{\partial}\ n]}\,c^m(\theta_R\Gamma^n\lambda_R) \tag{211}$$

but is a pure gauge, namely it is $Q_L + Q_R$ of:

$$-c^p c^q x_L^r B_{pqr\ ;\ [m\ \overleftarrow{\partial}\ n]}\,x_R^m(\theta_R\Gamma^n\lambda_R) + c^p x_L^q(\theta_L\Gamma^r\lambda_R)B_{pqr\ ;\ [m\ \overleftarrow{\partial}\ n]}\,c^m x_R^n \tag{212}$$

And the Q_{Lie} of this gives:

$$-c^p c^q c^r B_{pqr\ ;\ [m\ \overleftarrow{\partial}\ n]}\,x_R^m(\theta_R\Gamma^n\lambda_R) + c^p x_L^q(\theta_L\Gamma^r\lambda_R)B_{pqr\ ;\ [m\ \overleftarrow{\partial}\ n]}\,c^m c^n + \tag{213}$$

$$+c^p c^q x_L^r B_{pqr\ ;\ [m\ \overleftarrow{\partial}\ n]}\,c^m(\theta_R\Gamma^n\lambda_R) - c^p c^q(\theta_L\Gamma^r\lambda_L)B_{pqr\ ;\ [m\ \overleftarrow{\partial}\ n]}\,c^m x_R^n \tag{214}$$

The second row is $-(Q_L + Q_R)c^p c^q x_L^r B_{pqr\ ;\ mn} c^m x_R^n$. And the first row is equivalent, in the Maxwell cohomology, to the expression:

$$-c^p c^q c^r \mathcal{B}_{[pqrmn]}(dx_R^m \wedge dx_R^n - dx_L^m \wedge dx_L^n) \tag{215}$$

where $\mathcal{B}_{[pqrmn]}$ is defined in (151). This can be used to kill any class of the form:

$$c^p c^q c^r G_{[pqrst]}(dx_R^s \wedge dx_R^t - dx_L^s \wedge dx_L^t) \tag{216}$$

in $H^3(Q_{\text{Lie}}\ ,\ \text{SMaxw}_L \bigoplus \text{SMaxw}_R)$. The classes of the form:

$$c^p c^q c^r H_{[pqrst]}(dx_R^s \wedge dx_R^t + dx_L^s \wedge dx_L^t) \tag{217}$$

are not in the image of d_2. However, the d_2 of them is nonzero, giving an element of $\mathcal{E}_2^{5,0} = H^5(\mathbf{C}^{10})$ of the form $c^p c^q c^r c^s c^t H_{pqrst}$.

We conclude that $\mathcal{E}_3^{3,1} = 0$.

8. Action of the supersymmetry on the ghost number three vertices

In this section we will study the action of the supersymmetry on the ghost number three vertices. We will first act by the left supersymmetry on the element of the Maxwell–Dirac sector, and see that the result is some element of the Dirac–Dirac sector. Then we will act my the left supersymmetry on the Dirac–Dirac sector, which will bring us back to the Maxwell–Dirac sector. We will verify that the anticommutator of two supersymmetries is a translation.

8.1. *Left supersymmetry on the Maxwell–Dirac sector*

Let us consider an element of the Maxwell–Dirac sector, of the following form:

$$\Psi_m(x_R) c_n dx_L^n \wedge dx_L^m + \dots \tag{218}$$

where $\dots$ stand for elements of the lower degree in x_R (which have dependence on x_L). Let us act on it by the *left* supersymmetry with the parameter ϵ^α, which we will call S_ϵ. To evaluate the action of this supersymmetry, we will use the formulas from Section 6.1.3 of [12] (where S_ϵ was denoted $Q^{\mathcal{H}}_{\mathrm{Lie}}$, and ϵ^α was called ξ^α). We get the following element of the Dirac–Dirac sector:

$$-\frac{2}{3} \times \frac{1}{2}[\hat{c}, \Gamma^m]\epsilon\, \Psi_m + \dots \tag{219}$$

We observe:

$$\Gamma^j \frac{\partial}{\partial x_L^j}\left(\frac{1}{2}[\hat{x}_L, \Gamma^m]\epsilon\, \Psi_m\right) = \Gamma^j \frac{\partial}{\partial x_L^j}\left(\frac{9}{10}\hat{x}_L \Gamma^m \epsilon \Psi_m\right) \tag{220}$$

This implies that (219) gives the same cohomology class as:

$$-\frac{2}{3} \times \frac{9}{10}\hat{c}\Gamma^n \epsilon\, \Psi_n(x_R) + \dots \tag{221}$$

In notations of Section 5.1.2 we have:

$$R = -\frac{2}{3} \times \frac{9}{10}\Gamma^n \epsilon\, \Psi_n(x_R) \tag{222}$$

The obstacle to the triviality is:

$$\Gamma^m \partial_m R = -\frac{2}{3} \times \frac{9}{10}\Gamma^m \Gamma^n \epsilon\, \partial_m \Psi_n(x_R) = -\frac{2}{3} \times \frac{9}{10}\Gamma^{mn}\epsilon\, \partial_{[m}\Psi_{n]}(x_R) \tag{223}$$

(This is a bispinor: $(\Gamma^{mn}\epsilon)^\alpha\ (\partial_{[m}\Psi_{n]}(x_R))^{\dot\beta}$.)

8.2. *Left supersymmetry on the Dirac–Dirac sector*

We want to calculate the action of the supersymmetry with the parameter ϵ on the class:

$$\hat{c}R(x_R) + \dots \tag{224}$$

This is a bit tricky, because the leading term $\hat{c}R(x_R)$ does not contribute, and we have to analyze the subleading term proportional to x_L:

$$\begin{aligned}&\hat{c}R(x_R) + \\ &+ \frac{5}{6}\hat{c}x_L^k \partial_k R(x_R) - \frac{1}{6}\hat{x}_L c^k \partial_k R(x_R) + \frac{5}{6}(c \cdot x_L)\Gamma^k \partial_k R + \dots\end{aligned} \tag{225}$$

where ... stand for terms of the higher order in x_L. Again, we use the formulas from [12]. When acting by the supersymmetry with the parameter ϵ, we are getting the following element of the Maxwell–Dirac sector:

$$-\frac{3}{2}\epsilon\Gamma_{[n}\left(\frac{5}{6}\hat{c}\partial_{m]}R(x_R) - \frac{1}{6}\Gamma_{m]}c^k\partial_k R(x_R) + \frac{5}{6}c_{m]}\Gamma^k\partial_k R(x_R)\right) dx_L^n \wedge dx_L^m \tag{226}$$

This can be written as follows:

$$\begin{aligned} & -\frac{3}{2}\, c^k Y^l_{kmn}\partial_l R\, dx_L^n \wedge dx_L^m \\ & \texttt{where}\quad Y^l_{kmn} = -\frac{5}{6}\Gamma^l{}_{[n}\delta_{m]k} - \frac{5}{6}\Gamma_{k[n}\delta^l_{m]} - \frac{1}{6}\Gamma_{nm}\delta^l_k \end{aligned} \tag{227}$$

We can add $Q_{\text{Lie}}\left(\epsilon\Gamma_{nm}R dx_L^n \wedge dx_L^m\right)$ then we get:

$$\begin{aligned} & -\frac{3}{2}\, c^k \widetilde{Y}^l_{kmn}\partial_l R\, dx_L^n \wedge dx_L^m \\ & \texttt{where}\quad \widetilde{Y}^l_{kmn} = -\frac{5}{6}\Gamma^l{}_{[n}\delta_{m]k} - \frac{5}{6}\Gamma_{k[n}\delta^l_{m]} + \frac{5}{6}\Gamma_{nm}\delta^l_k \end{aligned} \tag{228}$$

Now $\widetilde{Y}^l_{[kmn]} = 0$ and $\widetilde{Y}^l_{mmn} = -5\Gamma^l{}_n$. Consider the following tensor:

$$Z^l_{kmn} = \frac{5}{18}\Gamma^l{}_{[n}\delta_{m]k} - \frac{5}{6}\Gamma_{k[n}\delta^l_{m]} + \frac{5}{6}\Gamma_{nm}\delta^l_k \tag{229}$$

It satisfies $Z^l_{[kmn]} = 0$ and $Z^l_{mmn} = 0$. Therefore the cohomology class does not change if we replace $\widetilde{Y}^l_{kmn}$ as follows:

$$\widetilde{Y}^l_{kmn} \mapsto \widetilde{Y}^l_{kmn} - Z^l_{kmn} = -\frac{10}{9}\Gamma^l{}_{[n}\delta_{m]k} \tag{230}$$

Indeed:

$$\begin{aligned} & c^k Z^l_{kmn}\partial_l R\, dx_L^n \wedge dx_L^m + \ldots = \\ & = c^m\left(\frac{\partial}{\partial x_L^m} - \frac{\partial}{\partial x_R^m}\right)\left(x_L^k Z^l_{kmn}\partial_l R\, dx_L^n \wedge dx_L^m + \ldots\right) \end{aligned} \tag{231}$$

We conclude that the supersymmetry with the parameter ϵ brings $\hat{c}R + \ldots$ to $-\frac{10}{9} \times \left(-\frac{3}{2}\right)\epsilon\Gamma_{ln}\partial^l R\, c_k dx^k \wedge dx^n$. When R is given by (222), we get:

$$-\epsilon\Gamma_{ln}\Gamma^j\epsilon\partial^l_R\Psi_j c_k dx^k \wedge dx^n = -(\epsilon\Gamma^l\epsilon)\,\frac{\partial}{\partial x^l_R}(\Psi_j(x_R)c_k dx_L^k \wedge dx_L^j) \tag{232}$$

This is in agreement with the fact that the anticommutator of two SUSY transformations is a translation.

8.3. Conclusion

Ghost number three vertices transform in the linearized Type IIB supergravity supermultiplet.

Acknowledgements

We would like to thank Nathan Berkovits for useful discussions. This work was supported in part by the Ministry of Education and Science of the Russian Federation under the project 14.740.11.0347 “Integrable and algebro-geometric structures in string theory and quantum field theory”, and in part by the RFBR grant 15-01-99504 “String theory and integrable systems”.

References

[1] A. Astashkevich, A. Belopolsky, String center-of-mass operator and its effect on BRST cohomology, Commun. Math. Phys. 186 (1997) 109–136, http://dx.doi.org/10.1007/BF02885675, arXiv:/hep-th/9511111.
[2] P.C. Nelson, Covariant insertion of general vertex operators, Phys. Rev. Lett. 62 (1989) 993, http://dx.doi.org/10.1103/PhysRevLett.62.993.
[3] A. Mikhailov, Cornering the unphysical vertex, J. High Energy Phys. 082 (2012), http://dx.doi.org/10.1007/JHEP11(2012)082, arXiv:1203.0677.
[4] B. Zwiebach, Closed string field theory: quantum action and the B–V master equation, Nucl. Phys. B 390 (1993) 33–152, http://dx.doi.org/10.1016/0550-3213(93)90388-6, arXiv:hep-th/9206084.
[5] B.H. Lian, G.J. Zuckerman, New perspectives on the BRST algebraic structure of string theory, Commun. Math. Phys. 154 (1993) 613–646, http://dx.doi.org/10.1007/BF02102111, arXiv:hep-th/9211072.
[6] N. Berkovits, P.S. Howe, Ten-dimensional supergravity constraints from the pure spinor formalism for the superstring, Nucl. Phys. B 635 (2002) 75–105, arXiv:hep-th/0112160.
[7] O. Chandia, A. Mikhailov, B.C. Vallilo, A construction of integrated vertex operator in the pure spinor sigma-model in $AdS_5 \times S^5$, J. High Energy Phys. 1311 (2013) 124, http://dx.doi.org/10.1007/JHEP11(2013)124, arXiv:1306.0145.
[8] O.A. Bedoya, L. Bevilaqua, A. Mikhailov, V.O. Rivelles, Notes on beta-deformations of the pure spinor superstring in $AdS(5) \times S(5)$, Nucl. Phys. B 848 (2011) 155–215, http://dx.doi.org/10.1016/j.nuclphysb.2011.02.012, arXiv:1005.0049.
[9] A. Mikhailov, Pure spinors in AdS and Lie algebra cohomology, Lett. Math. Phys. 104 (2014) 1201–1233, http://dx.doi.org/10.1007/s11005-014-0705-2, arXiv:1207.2441.
[10] R.L. Jusinskas, On the field-antifield (a)symmetry of the pure spinor superstring, arXiv:1510.05268.
[11] N. Berkovits, N. Nekrasov, Multiloop superstring amplitudes from non-minimal pure spinor formalism, J. High Energy Phys. 0612 (2006) 029, http://dx.doi.org/10.1088/1126-6708/2006/12/029, arXiv:hep-th/0609012.
[12] A. Mikhailov, Symmetries of massless vertex operators in AdS(5) x S**5, J. Geom. Phys. (2011), http://dx.doi.org/10.1017/j.geomphys.2011.09.002, arXiv:0903.5022.